Population Growth Models: Geometric Growth

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Purpose of population models

- Project into the future the current demography (e.g., survivorship and reproduction)
- Gauge the potential (or lack) for a population to increase
- Determine the consequences of changes in the current demography
Population Models in General

- Observables: \( N \) or \( N(\text{age}) \) or \( N(\text{stage}) \)
- Project population size \( N \) as a function of time \( t \)
- Projection in terms of fundamental parameters:
  - describing demographic events in an individual’s life
  - e.g., \( \text{Pr}(\text{birth}), \text{Pr}(\text{death}) \)
- enable understanding of how demographic vital rates affect the whole population
Projection versus Prediction

- No population experiences unlimited resources
- Yet, all populations have potential for exponential growth
- Projections describe potential, not what is actually predicted to occur
  - analogy: a speedometer projects potential travel only
Geometric Growth Models

General motivation

- Sequence of population sizes through time
  \[ N_t, N_{t+1}, N_{t+2}, \ldots \]

- Change from one time to next
  - increases due to births during period
  - decreases due to deaths during period
  - increases due to immigrants during period
  - decreases due to emigrants during period
Mathematical Formulation

- Population size after an interval of time

\[ N_{t+1} = N_t + B - D + I - E \]  

- \( B, D \): birth, death
- \( I, E \): immigration, emigration

- Change in population size

\[ \Delta N = N_{t+1} - N_t = B - D + I - E \]  

- Closed versus open populations
Geometric Growth Model: Assumptions

- Closed population: \( I = E = 0 \)
- Constant per capita birth (\( b \)) and death (\( d \)) rates
  - \( B = bN \)
  - \( D = dN \)
Geometric Growth Model: Assumptions

- Closed population: \( I = E = 0 \)
- Constant per captita birth (\( b \)) and death (\( d \)) rates
  - \( B = bN \)
  - \( D = dN \)
- Unlimited resources
- No genetic structure
  - \( b \) and \( d \) identical for all individuals regardless of genotype
- No age- or size-structure
  - \( b \) and \( d \) identical for all individuals regardless of size, age, \ldots
- No time lags
  - birth and death depend on current population only

\[ N_{t+1} = N_t + B - D \]  
\[ = N_t + \left( \frac{B}{N_t} - \frac{D}{N_t} \right) \cdot N_t \]  
\[ = N_t + (b - d) \cdot N_t \]  
\[ = (1 + (b - d)) \cdot N_t \]  
\[ = (1 + R_t) \cdot N_t \]  
\[ = \lambda_t \cdot N_t \]  
\[ \lambda_t = \frac{N_{t+1}}{N_t} \]
Geometric Population Growth: Change in Population Size

\[
\Delta N = N_{t+1} - N_t
\]
\[
= (1 + R_t)N_t - N_t
\]
\[
= N_t + R_tN_t - N_t
\]
\[
= R_tN_t
\]
\[
R_t = \frac{\Delta N}{N_t}
\]
Finite Rate of Increase: $\lambda$

\[ N_{t+1} = \lambda_t N_t \]  
(16)

\[ \lambda_t = \frac{N_{t+1}}{N_t} \]  
(17)

- population increase: $\lambda > 1$
- population stable: $\lambda = 1$
- population decrease: $\lambda < 1$
Projection of Population Size

Assume a constant value of $\lambda$: i.e., $\lambda_t = \lambda$

\[
\begin{align*}
N_1 &= \lambda N_0 \quad \text{(18)} \\
N_2 &= \lambda N_1 \\
      &= \lambda (\lambda N_0) \\
      &= \lambda^2 N_0 \quad \text{(20)} \\
N_t &= \lambda N_{t-1} \quad \text{(22)} \\
      &= \lambda (\lambda N_{t-2}) \\
      &= \lambda (\lambda (\lambda N_{t-3})) \\
      &= \lambda^t N_0 \quad \text{(25)}
\end{align*}
\]
Geometric Population Model: Doubling Time

How long does it take the population to double in size? That is, how long does it take the population to change from $N_0$ to $2N_0$?

$N_t = \lambda^t N_0$ \hfill (26)

$2N_0 = \lambda^t N_0$ \hfill (27)
Geometric Population Model: Doubling Time

How long does it take the population to double in size? That is, how long does it take the population to change from $N_0$ to $2N_0$?

\[ N_t = \lambda^t N_0 \quad (26) \]
\[ 2N_0 = \lambda^t N_0 \quad (27) \]
\[ 2 = \lambda^t \quad (28) \]
How long does it take the population to double in size?
That is, how long does it take the population to change from $N_0$ to $2N_0$?

\begin{align}
N_t &= \lambda^t N_0 \\
2N_0 &= \lambda^t N_0 \\
2 &= \lambda^t \\
\ln(2) &= \ln(\lambda^t) \\
\ln(2) &= t \cdot \ln(\lambda) \\
t &= \frac{\ln(2)}{\ln(\lambda)}
\end{align}
Geometric Population Model: Half Life

How long does it take the population to become half as large in size? That is, how long does it take the population to change from $N_0$ to $\frac{1}{2}N_0$?

\[
N_t = \lambda^t N_0 \quad (32)
\]
\[
\frac{1}{2}N_0 = \lambda^t N_0 \quad (33)
\]
Geometric Population Model: Half Life

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\[ N_t = \lambda^t N_0 \]  
\[ \frac{1}{2} N_0 = \lambda^t N_0 \]  
\[ \frac{1}{2} = \lambda^t \]
Geometric Population Model: Half Life

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\frac{1}{2} N_0 = \lambda^t N_0 \quad (33)
\]
\[
\frac{1}{2} = \lambda^t \quad (34)
\]
\[
\ln\left(\frac{1}{2}\right) = \ln(\lambda^t) \quad (35)
\]
Geometric Population Model: Half Life

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\frac{1}{2} = \lambda^t \quad \text{(34)}
\]
\[
\ln\left(\frac{1}{2}\right) = \ln(\lambda^t) \quad \text{(35)}
\]
\[
-\ln(2) = t \cdot \ln(\lambda) \quad \text{(36)}
\]
\[
t = -\frac{\ln(2)}{\ln(\lambda)} \quad \text{(37)}
\]
Geometric Population Model

- Quantitative description of how a population changes size as time progresses
- Depends directly on the finite rate of increase, $\lambda$
- $\lambda$ in turn depends on the per capita rates of birth and death (through their difference only)
- $\lambda$ measures the rate of increase
- $\lambda$ measures the potential for a population to grow
- Questions that can be answered:
  - Is the population increasing, decreasing, or stable?
  - What is the potential for the population to increase?
  - How long does it take for the population to change by a certain amount?
  - How will the answers change if the vital rates ($b$ and $d$) change?