

Dances between continuous and discrete: Euler's summation formula

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1 Introduction

Leonhard Euler (1707–1783) discovered his powerful “summation formula” in the early 1730s. He used it in 1735 to compute the first 20 decimal places for the precise sum of all the reciprocal squares — a number mathematicians had competed to determine ever since the surprising discovery that the alternating sum of reciprocal odd numbers is $\pi/4$. This reciprocal squares challenge was called the “Basel Problem”, and Euler achieved his 20-place approximation to it using only a few terms from his diverging summation formula. In contrast, if sought as a simple partial sum of the original slowly converging series, such accuracy would require an overwhelming more than 10^{20} terms. With his approximation, Euler was probably convincing himself that the sum was exactly $\pi^2/6$, which spurred his first solution of the celebrated Basel Problem in the same year [7, v. 14, v. 16, section 2, pp. VIIff][18, 26].

We are left in awe that just a few terms of a diverging formula can so closely approximate this sum. Paradoxically, Euler's formula, even though it usually diverges, provides breathtaking acceleration of approximations for partial and infinite sums of many slowly converging or diverging series. My goal here is to explore Euler's own mature view of the summation formula and a few of his more diverse applications, largely in his own words from his

book *Institutiones Calculi Differentialis (Foundations of Differential Calculus)* of 1755, never before available in English. I hope that readers will be equally impressed at some of his other applications.

In the *Institutiones*, Euler connected his summation formula to Bernoulli numbers, and proved Bernoulli's conjectured sums of powers formulas. He also applied the formula to harmonic partial sums and the related gamma constant, and to sums of logarithms, thereby approximating large factorials (Stirling's asymptotic approximation) and binomial coefficients with ease. He even made an approximation of π that he himself commented was hard to believe so accurate for so little work. Euler was a wizard at finding these connections, at demonstrating patterns by generalizable example, at utilizing his summation formula only "until it begins to diverge", and at determining the relevant "Euler-Maclaurin constant" in each application of the summation formula. His work also inaugurated study of the zeta function [2, 24]. Euler's accomplishments throughout this entire arena are discussed from different points of view in many modern books [5][11, pp. 119–136][12, II.10][13, ch. XIII][15, p. 197f][19, ch. XIV] [26, p. 184, 257–285][28, p. 338f].

Euler included all of these discoveries and others in beautifully unified form in Part Two¹ of the *Institutiones* [7, vol. 10][8], portions of which I have translated for teaching an undergraduate course based on original sources [20, 21, 22], and for selective inclusion² in a companion book built around annotated sources [18]. Others have also used this material in teaching honors calculus. More expansively, our book chapter *The Bridge between the Continuous and the Discrete* [23] follows the entwining of the quest for formulas for sums of numerical powers with integration, through original sources by Archimedes, Fermat, Pascal, Jakob Bernoulli, and finally from Euler's *Institutiones*.

I will first discuss the Basel Problem, and briefly outline the progression of earlier ideas and sources that led to the connection in Euler's work between sums of powers and the Basel Problem. Then I will illustrate a few of Euler's achievements with the summation formula via selected translations. I present Euler's derivation of the formula, discuss his analysis of the resulting Bernoulli numbers, show his application to sums of reciprocal squares, to large factorials and binomial coefficients, and mention other applications. I will also raise and explore the question of whether large factorials can be

¹Part One has recently appeared in English translation [9], but not Part Two.

²See [10] for my most extensive translation from Euler's Part Two (albeit more lightly annotated).

determined uniquely from Euler’s formula. See [18, 23] for a fuller exposition and treatment of much of this material.

2 The Basel Problem

In the 1670s, James Gregory (1638–1675) and Gottfried Leibniz (1646–1716) discovered that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4},$$

as essentially had the mathematicians of Kerala in southern India two centuries before [16, pp. 493ff,527]. Since, aside from geometric series, very few infinite series then had a known sum, this remarkable result enticed Leibniz and the Bernoulli brothers Jakob (1654–1705) and Johann (1667–1748) to seek sums of other series, particularly the reciprocal squares

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = ?,$$

a problem first raised by Pietro Mengoli (1626–1686) in 1650. Jakob expressed his eventual frustration at its elusive nature in the comment “*If someone should succeed in finding what till now withstood our efforts and communicate it to us, we shall be much obliged to him*” [28, p. 345].

Euler proved that the sum is exactly $\pi^2/6$, in part by a broadening of the context to produce his “summation formula” for $\sum_{i=1}^n f(i)$, with n possibly infinite. His new setting thus encompassed both the Basel Problem, $\sum_{i=1}^{\infty} 1/i^2$, and the quest for closed formulas for sums of powers, $\sum_{i=1}^n i^k \approx \int_0^n x^k dx$, which had been sought since antiquity for area and volume investigations. The summation formula helped Euler resolve both questions. This is a fine pedagogical illustration of how generalization and abstraction can lead to the combined solution of seemingly independent problems.

3 Sums of powers and Euler’s summation formula: historically interlocked themes

Our story (told more completely in [18, 23]) begins in ancient times with the Greek discrete approximations used to obtain continuous areas and volumes by the method of exhaustion. The Pythagoreans (sixth c. B.C.E.) knew that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2},$$

and Archimedes (third c. B.C.E.) proved an equivalent to our modern formula

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

which he applied to deduce the area inside a spiral [1, Spirals]:

The area bounded by the first turn of the spiral and the initial line is equal to one-third of the first circle.

Summing yet higher powers was key to finding other areas and volumes, and one finds the formula for a sum of cubes in work of Nicomachus of Gerasa (first c. B.C.E.), Āryabhaṭa in India (499 C.E.), and al-Karajī in the Arab world (c. 1000) [4][14, p. 68f][16, p. 212f,251ff]. The first evidence of a general relationship between various exponents is in the further Arabic work of Abū ‘Alī al-Ḥasan ibn al-Haytham (965–1039), who needed a formula for sums of fourth powers to find the volume of a paraboloid of revolution. He discovered a doubly recursive relationship between varying exponents [16, p. 255f].

By the mid-seventeenth century Pierre de Fermat (1601–1665) and Blaise Pascal (1623–1662) had realized the general connection between the figurate (equivalently binomial coefficient) numbers and sums of powers, motivated by the drive to determining areas under “higher parabolas” (i.e., $y = x^k$) [16, p. 481ff]. Fermat called the sums of powers challenge “*what is perhaps the most beautiful problem of all arithmetic*”, and claimed a recursive solution using figurate numbers. Pascal used binomial expansions and telescoping sums to obtain the first simply recursive relationship between sums of powers for varying exponents [4].

Jakob Bernoulli, during his work in the nascent field of probability, was the first to conjecture a general pattern in sums of powers formulas, simultaneously introducing the Bernoulli numbers into mathematics³. Within his posthumous book of 1713, *The Art of Conjecturing* [3, vol. 3, pp. 164–167], appears a section on *A Theory of Permutations and Combinations*. Here one finds him first list the formulas for *Sums of Powers* up to exponent ten (using the notation \int for the discrete sum from 1 to n), and then claim a pattern, to wit⁴:

³The evidence suggests that around the same time, Takakazu Seki (1642?–1708) in Japan also discovered the same numbers [25, 27].

⁴There is an error in the original published Latin table of sums of powers formulas. The last coefficient in the formula for $\int n^9$ should be $-\frac{3}{20}$, not $-\frac{1}{12}$; we have corrected this here.

$$\begin{aligned}
\int n &= \frac{1}{2}nn + \frac{1}{2}n. \\
\int nn &= \frac{1}{3}n^3 + \frac{1}{2}nn + \frac{1}{6}n. \\
\int n^3 &= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}nn. \\
\int n^4 &= \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 * -\frac{1}{30}n. \\
\int n^5 &= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 * -\frac{1}{12}nn. \\
\int n^6 &= \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 * -\frac{1}{6}n^3 * +\frac{1}{42}n. \\
\int n^7 &= \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 * -\frac{7}{24}n^4 * +\frac{1}{12}nn. \\
\int n^8 &= \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 * -\frac{7}{15}n^5 * +\frac{2}{9}n^3 * -\frac{1}{30}n. \\
\int n^9 &= \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 * -\frac{7}{10}n^6 * +\frac{1}{2}n^4 * -\frac{3}{20}nn. \\
\int n^{10} &= \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 * -1n^7 * +1n^5 * -\frac{1}{2}n^3 * +\frac{5}{66}n.
\end{aligned}$$

Indeed, a pattern can be seen in the progressions herein which can be continued by means of this rule: Suppose that c is the value of any power; then the sum of all n^c or

$$\begin{aligned}
\int n^c &= \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c \cdot c - 1 \cdot c - 2}{2 \cdot 3 \cdot 4}Bn^{c-3} \\
&+ \frac{c \cdot c - 1 \cdot c - 2 \cdot c - 3 \cdot c - 4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}Cn^{c-5} \\
&+ \frac{c \cdot c - 1 \cdot c - 2 \cdot c - 3 \cdot c - 4 \cdot c - 5 \cdot c - 6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}Dn^{c-7} \dots,
\end{aligned}$$

where the value of the power n continues to decrease by two until it reaches n or nn . The uppercase letters A, B, C, D , etc., in order, denote the coefficients of the final term of $\int nn, \int n^4, \int n^6, \int n^8$, etc., namely

$$A = \frac{1}{6}, B = -\frac{1}{30}, C = \frac{1}{42}, D = -\frac{1}{30}.$$

These coefficients are such that, when arranged with the other coefficients of the same order, they add up to unity: so, for D , which we said signified $-\frac{1}{30}$,

we have

$$\frac{1}{9} + \frac{1}{2} + \frac{2}{3} - \frac{7}{15} + \frac{2}{9}(+D) - \frac{1}{30} = 1.$$

At this point we modern readers could conceivably exhibit great retrospective prescience, anticipate Euler's broader context of $\sum_{i=1}^n f(i)$, for which Bernoulli's claimed summation formula above engages test functions of the form $f(x) = x^c$, and venture a rash generalization:

$$\sum_{i=1}^n f(i) \approx C + \int^n f(x)dx + \frac{f(n)}{2} + A\frac{f'(n)}{2!} + B\frac{f'''(n)}{4!} + \dots .$$

This formula is what Euler discovered in the early 1730s (although he was apparently unaware of Bernoulli's claim until later). Euler's summation formula captures the delicate details of the general connection between integration and discrete summation, and subsumes and resolves the two-thousand year old quest for sums of powers formulas as a simple special case. In what follows I will focus on just a few highlights from Euler.

4 The Basel Problem and the summation formula

"Euler calculated without any apparent effort, just as men breathe, as eagles sustain themselves in the air.", Arago. [28, p. 354]

Around the year 1730, the 23-year old Euler, along with his frequent correspondents Christian Goldbach (1690–1764) and Daniel Bernoulli (1700–1782), developed ways to find increasingly accurate fractional or decimal estimates for the sum of the reciprocal squares. But highly accurate estimates were challenging, since the series converges very slowly. They were likely trying to guess the exact value of the sum, hoping to recognize that their approximations hinted something familiar, perhaps involving π , like Leibniz's series which had summed to $\pi/4$. Euler hit gold with the discovery of his summation formula. One of its first major uses was in a paper submitted to the St. Petersburg Academy of Sciences on the 13th of October, 1735, in which he approximated the sum correctly to twenty decimal places. Only seven and a half weeks later Euler astonished his contemporaries with another paper, solving the famous Basel Problem completely, by demonstrating with a completely different method that the precise sum

of the series is $\pi^2/6$: “Now, however, quite unexpectedly, I have found an elegant formula for $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}$, depending upon the quadrature of the circle [i.e., upon π]” [26, p. 261]. Johann Bernoulli reacted “And so is satisfied the burning desire of my brother [Jakob] who, realizing that the investigation of the sum was more difficult than anyone would have thought, openly confessed that all his zeal had been mocked. If only my brother were alive now” [28, p. 345].

Much of Euler’s *Institutiones Calculi Differentialis*, written two decades later, is focused on the relationship between differential calculus and infinite series, unifying his many discoveries in a single exposition. He devotes Chapters 5 and 6 of Part Two to the summation formula and a treasure trove of applications. In Chapter 5 Euler derives his summation formula, analyzes the generating function for Bernoulli numbers in terms of transcendental functions, derives several properties of Bernoulli numbers, shows that they grow supergeometrically, proves Bernoulli’s formulas for sums of powers, and finds the exact sums of all infinite series of reciprocal even powers in terms of Bernoulli numbers. Chapter 6 applies the summation formula to approximate harmonic partial sums and the associated “Euler” constant γ , sums of reciprocal powers, π , and sums of logarithms, leading to approximations for large factorials and binomial coefficients.

I will guide the reader through glimpses of just a few key passages from the translation. The reader may find more background, annotation, and exercises in [18], or explore my more extensive translation on the web [10]. The passages displayed below contain Euler’s derivation, the relation to Bernoulli numbers, application to reciprocal squares, and to sums of logarithms, large factorials, and binomials, with mention of other omitted passages. Each application uses the summation formula to a fundamentally different purpose. The complete glory of Euler’s chapters is still available only in the original Latin [7, vol. 10] or an old German translation [8] (poorly printed in Fraktur); I cannot encourage the reader too strongly to revel in the beauty of the original.

5 Euler’s derivation

Euler’s derivation of the summation formula rests on two ideas. First, the use of what we now call Taylor series from calculus, combined with adeptness with summations, to relate the sum of the values of a function at finitely many successive integers to similar sums involving the derivatives of the function. It may seem that this just makes things more complicated, but

his second idea will remedy this.

Leonhard Euler, from
 Foundations of Differential Calculus
 Part Two, Chapter 5
 On Finding Sums of Series from the General Term

105. Consider a series whose general term, belonging to the index x , is y , and whose preceding term, with index $x - 1$, is v ; because v arises from y , when x is replaced by $x - 1$, one has⁵

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \frac{d^4y}{24dx^4} - \frac{d^5y}{120dx^5} + \text{etc.}$$

If y is the general term of the series

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & \cdots & x-1 & x \\ a & + & b & + & c & + & d & + & \cdots & + & v & + & y \end{array}$$

and if the term belonging to the index 0 is A , then v , as a function of x , is the general term of the series

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & \cdots & x \\ A & + & a & + & b & + & c & + & d & + & \cdots & + & v \end{array},$$

so if Sv denotes the sum of this series, then $Sv = Sy - y + A$.

106. Because

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \text{etc.},$$

one has, from the preceding,

$$Sv = Sy - S\frac{dy}{dx} + S\frac{ddy}{2dx^2} - S\frac{d^3y}{6dx^3} + S\frac{d^4y}{24dx^4} - \text{etc.},$$

⁵Euler expresses the value v of his function at $x - 1$ in terms of its value y at x and the values of all its derivatives, also implicitly evaluated at x . This uses Taylor series with increment -1 . Of course it seems that he is tacitly assuming that this all makes sense, i.e., that his function is infinitely differentiable, and perhaps also that the Taylor series converges and equals its intended value. Note also that the symbols x and y are being used, respectively, to indicate the final value of an integer index and the final value of the function evaluated there, and also more generally as a variable and function of that variable. Today we would find this much too confusing to dare write this way.

and, because $Sv = Sy - y + A$,

$$y - A = S \frac{dy}{dx} - S \frac{ddy}{2dx^2} + S \frac{d^3y}{6dx^3} - S \frac{d^4y}{24dx^4} + \text{etc.},$$

or equivalently

$$S \frac{dy}{dx} = y - A + S \frac{ddy}{2dx^2} - S \frac{d^3y}{6dx^3} + S \frac{d^4y}{24dx^4} - \text{etc.}$$

Thus if one knows the sums of the series, whose general terms are $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, etc., one can obtain the summative term of the series whose general term is $\frac{dy}{dx}$. The constant A must then be such that the summative term $S \frac{dy}{dx}$ disappears when $x = 0$...

Euler next applies this equation recursively, in §107–108, to demonstrate how one can obtain individual sums of powers formulas, since in these cases the derivatives will eventually vanish. He then continues with his second idea. The result is the summation formula.

109. ... if one sets $\frac{dy}{dx} = z$, then

$$Sz = \int z dx + \frac{1}{2} S \frac{dz}{dx} - \frac{1}{6} S \frac{ddz}{dx^2} + \frac{1}{24} S \frac{d^3z}{dx^3} - \text{etc.},$$

adding to it a constant value such that when $x = 0$, the sum Sz also vanishes.

...

110. But if in the expressions above one substitutes the letter z in place of y , or if one differentiates the preceding equation, which yields the same, one obtains

$$S \frac{dz}{dx} = z + \frac{1}{2} S \frac{ddz}{dx^2} - \frac{1}{6} S \frac{d^3z}{dx^3} + \frac{1}{24} S \frac{d^4z}{dx^4} - \text{etc.};$$

but using $\frac{dz}{dx}$ in place of y one obtains

$$S \frac{ddz}{dx^2} = \frac{dz}{dx} + \frac{1}{2} S \frac{d^3z}{dx^3} - \frac{1}{6} S \frac{d^4z}{dx^4} + \frac{1}{24} S \frac{d^5z}{dx^5} - \text{etc.}$$

... and so forth indefinitely...

111. Now when these values for $S \frac{dz}{dx}$, $S \frac{ddz}{dx^2}$, $S \frac{d^3z}{dx^3}$ are successively substituted in the expression

$$Sz = \int z dx + \frac{1}{2} S \frac{dz}{dx} - \frac{1}{6} S \frac{ddz}{dx^2} + \frac{1}{24} S \frac{d^3z}{dx^3} - \text{etc.},$$

one finds an expression for Sz , composed of the terms $\int z dx$, z , $\frac{dz}{dx}$, $\frac{ddz}{dx^2}$, $\frac{d^3z}{dx^3}$ etc., whose coefficients are easily obtained as follows. One sets

$$Sz = \int z dx + \alpha z + \frac{\beta dz}{dx} + \frac{\gamma ddz}{dx^2} + \frac{\delta d^3z}{dx^3} + \frac{\varepsilon d^4z}{dx^4} + \text{etc.},$$

and substitutes for these terms the values they have from the previous series, yielding

$$\begin{aligned} \int z dx &= Sz - \frac{1}{2}S\frac{dz}{dx} + \frac{1}{6}S\frac{ddz}{dx^2} - \frac{1}{24}S\frac{d^3z}{dx^3} + \frac{1}{120}S\frac{d^4z}{dx^4} - \text{etc.} \\ \alpha z &= +\alpha S\frac{dz}{dx} - \frac{\alpha}{2}S\frac{ddz}{dx^2} + \frac{\alpha}{6}S\frac{d^3z}{dx^3} - \frac{\alpha}{24}S\frac{d^4z}{dx^4} + \text{etc.} \\ \frac{\beta dz}{dx} &= \beta S\frac{ddz}{dx^2} - \frac{\beta}{2}S\frac{d^3z}{dx^3} + \frac{\beta}{6}S\frac{d^4z}{dx^4} - \text{etc.} \\ \frac{\gamma ddz}{dx^2} &= \gamma S\frac{d^3z}{dx^3} - \frac{\gamma}{2}S\frac{d^4z}{dx^4} + \text{etc.} \\ \frac{\delta d^3z}{dx^3} &= \delta S\frac{d^4z}{dx^4} - \text{etc.} \end{aligned}$$

etc.

Since these values, added together, must produce Sz , the coefficients α , β , γ , δ etc. are ...

112. ...

$$\begin{aligned} \alpha &= \frac{1}{2}, \beta = \frac{\alpha}{2} - \frac{1}{6} = \frac{1}{12}, \gamma = \frac{\beta}{2} - \frac{\alpha}{6} + \frac{1}{24} = 0, \\ \delta &= \frac{\gamma}{2} - \frac{\beta}{6} + \frac{\alpha}{24} - \frac{1}{120} = -\frac{1}{720}, \varepsilon = \frac{\delta}{2} - \frac{\gamma}{6} + \frac{\beta}{24} - \frac{\alpha}{120} + \frac{1}{720} = 0 \text{ etc.}, \end{aligned}$$

and if one continues in this fashion one finds that alternating terms vanish.

6 Connection to Bernoulli numbers and sums of powers

Before Euler shows how to apply his summation formula to derive new results for various choices of the function z , and of the initial and final indices in a summation, he studies intensively in §112–120 the coefficients $\alpha, \beta, \gamma, \dots$, and discovers their properties and intimate connections with familiar functions. In particular, he shows how their generating function relates directly to the transcendental functions of calculus. Euler proves that alternate coefficients vanish, and that those remaining alternate in sign. For this he investigates a power series solution to the nonlinear differential

equation which the cotangent function satisfies by dint of its derivative formula. The differential equation produces quadratic recursive formulas for the coefficients. Euler also explores the number theoretic properties of the coefficients, including the growth and prime factorizations of their numerators and denominators, some of which we will see below.

Caution: In the process of distilling the summation formula in terms of Bernoulli numbers, Euler switches the meaning of the Greek letters $\alpha, \beta, \gamma, \delta, \dots$, and the formula now takes revised form:

121. ... If one finds the values of the [redefined] letters $\alpha, \beta, \gamma, \delta$, etc. according to this rule, which entails little difficulty in calculation, then one can express the summative term of any series, whose general term = z corresponding to the index x , in the following fashion:

$$Sz = \int zdx + \frac{1}{2}z + \frac{\alpha dz}{1 \cdot 2 \cdot 3 \cdot dx} - \frac{\beta d^3 z}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 dx^3} + \frac{\gamma d^5 z}{1 \cdot 2 \cdots 7 dx^5} - \frac{\delta d^7 z}{1 \cdot 2 \cdots 9 dx^7} + \frac{\varepsilon d^9 z}{1 \cdot 2 \cdots 11 dx^9} - \frac{\zeta d^{11} z}{1 \cdot 2 \cdots 13 dx^{11}} + \text{etc.} \dots$$

122. These numbers have great use throughout the entire theory of series. First, one can obtain from them the final terms in the sums of even powers, for which we noted above (in §63 of part one) that one cannot obtain them, as one can the other terms, from the sums of earlier powers. For the even powers, the last terms of the sums are products of x and certain numbers, namely for the 2nd, 4th, 6th, 8th, etc., $\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}$ etc. with alternating signs. But these numbers arise from the values of the letters $\alpha, \beta, \gamma, \delta$, etc., which we found earlier, when one divides them by the odd numbers 3, 5, 7, 9, etc. These numbers are called the Bernoulli numbers after their discoverer Jakob Bernoulli, and they are

$\frac{\alpha}{3} = \frac{1}{6} = \mathfrak{A}$	$\frac{\iota}{19} = \frac{43867}{798} = \mathfrak{J}$
$\frac{\beta}{5} = \frac{1}{30} = \mathfrak{B}$	$\frac{\chi}{21} = \frac{174611}{330} = \mathfrak{K} = \frac{283 \cdot 617}{330}$
$\frac{\gamma}{7} = \frac{1}{42} = \mathfrak{C}$	$\frac{\lambda}{23} = \frac{854513}{138} = \mathfrak{L} = \frac{11 \cdot 131 \cdot 593}{2 \cdot 3 \cdot 23}$
$\frac{\delta}{9} = \frac{1}{30} = \mathfrak{D}$	$\frac{\mu}{25} = \frac{236364091}{2730} = \mathfrak{M}$
$\frac{\varepsilon}{11} = \frac{5}{66} = \mathfrak{E}$	$\frac{\nu}{27} = \frac{8553103}{6} = \mathfrak{N} = \frac{13 \cdot 657931}{6}$
$\frac{\zeta}{13} = \frac{691}{2730} = \mathfrak{F}$	$\frac{\xi}{29} = \frac{23749461029}{870} = \mathfrak{O}$
$\frac{\eta}{15} = \frac{7}{6} = \mathfrak{G}$	$\frac{\pi}{31} = \frac{8615841276005}{14322} = \mathfrak{P}$
$\frac{\theta}{17} = \frac{3617}{510} = \mathfrak{H}$	etc.

Euler’s very first application of the Bernoulli numbers, in §124–125, is to solve a problem dear to his heart, determining the precise sums of all infinite series of reciprocal even powers. His result (using today’s notation $\mathfrak{A} = B_2$, $\mathfrak{B} = -B_4$, $\mathfrak{C} = B_6$, ...) is that

$$\sum_{i=1}^{\infty} \frac{1}{i^{2n}} = \frac{(-1)^{n+1} B_{2n} 2^{2n-1}}{(2n)!} \pi^{2n} \text{ for all } n \geq 1.$$

Since these sums approach one as n grows, he also obtains, in §129, an asymptotic understanding of how Bernoulli numbers grow:

$$\frac{B_{2n+2}}{B_{2n}} \approx -\frac{(2n+2)(2n+1)}{4\pi^2} \approx -\frac{n^2}{\pi^2}.$$

Thus he comments that they “*form a highly diverging sequence, which grows more strongly than any geometric sequence of growing terms*”.

This completes Euler’s analysis of the Bernoulli numbers themselves. Now he is ready to turn his summation formula towards applications. He ends Chapter 5 with applications in which the summation formula is finite (§131), including that of a pure power function, which proves Bernoulli’s conjectured sums of powers formulas (§132).

7 “Until it begins to diverge”

Chapter 6 uses the summation formula to make approximations even when it diverges, which it does in almost all interesting applications.

Part Two, Chapter 6

On the summing of progressions via infinite series

140. The general expression, that we found in the previous chapter for the summative term of a series, whose general term corresponding to the index x is z , namely

$$Sz = \int z dx + \frac{1}{2}z + \frac{\mathfrak{A}dz}{1 \cdot 2dx} - \frac{\mathfrak{B}d^3z}{1 \cdot 2 \cdot 3 \cdot 4dx^3} + \frac{\mathfrak{C}d^5z}{1 \cdot 2 \cdot \dots \cdot 6dx^5} - \text{etc.},$$

actually serves to determine the sums of series, whose general terms are integral rational functions⁶ of the index x , because in these cases one eventually arrives

⁶By this he means polynomials.

at vanishing differentials. On the other hand, if z is not such a function of x , then the differentials continue without end, and there results an infinite series that expresses the sum of the given series up to and including the term whose index = x . The sum of the series, continuing without end, is thus given by taking $x = \infty$, and one finds in this way another infinite series equal to the original. ...

142. Since when a constant value is added to the sum, so that it vanishes when $x = 0$, the true sum is then found when x is any other number, then it is clear that the true sum must likewise be given, whenever a constant value is added that produces the true sum in any particular case. Thus suppose it is not obvious, when one sets $x = 0$, what value the sum assumes and thus what constant must be used; one can substitute other values for x , and through addition of a constant value obtain a complete expression for the sum. Much will become clear from the following.

For a particular choice of antiderivative $\int z dx$, the constant which needs determining here is today called the “Euler-Maclaurin constant” for the function z and a chosen antiderivative $\int z dx$.

There follow Euler’s §142a–144, in which he makes the first application of his summation formula to an infinite series, the diverging harmonic series $\sum_{i=1}^{\infty} 1/i$. For this particular series, the Euler-Maclaurin constant in his summation formula will be the limiting difference between $\sum_{i=1}^x 1/i$ and $\ln x$. Today we call this particular number just “Euler’s constant”, and denote it by γ . It is arguably the third most important special constant in mathematics after π and e . Euler shows how to extract from the summation formula an approximation of γ accurate to 15 places, and then easily obtains the sum of the first thousand terms of the diverging harmonic series to 13 places (see [10]). In fact it is clear from what he writes that one could use his approach to approximate γ to whatever accuracy desired, and then apply the summation formula to find the value of arbitrarily large finite harmonic sums to that same accuracy. I will discuss in a moment the conundrum that he can obtain arbitrarily accurate approximations for the Euler-Maclaurin constant of a function and a chosen antiderivative from a diverging summation formula!

We continue on to see exactly how Euler applies the summation formula to that old puzzle, the Basel Problem.

148. After considering the harmonic series we wish to turn to examining the series of reciprocals of the squares, letting

$$s = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{xx}.$$

Since the general term of this series is $z = \frac{1}{xx}$, then $\int z dx = \frac{-1}{x}$, the differentials of z are

$$\frac{dz}{2dx} = -\frac{1}{x^3}, \quad \frac{ddz}{2 \cdot 3dx^2} = \frac{1}{x^4}, \quad \frac{d^3z}{2 \cdot 3 \cdot 4dx^3} = -\frac{1}{x^5} \quad \text{etc.},$$

and the sum is

$$s = C - \frac{1}{x} + \frac{1}{2xx} - \frac{\mathfrak{A}}{x^3} + \frac{\mathfrak{B}}{x^5} - \frac{\mathfrak{C}}{x^7} + \frac{\mathfrak{D}}{x^9} - \frac{\mathfrak{E}}{x^{11}} + \text{etc.},$$

where the added constant C is determined from one case in which the sum is known. We therefore wish to set $x = 1$. Since then $s = 1$, one has

$$C = 1 + 1 - \frac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \mathfrak{E} - \text{etc.},$$

but this series alone does not give the value of C , since it diverges strongly.

On the face of it, these formulas seem both absurd and useless. The expression Euler obtains for the Euler-Maclaurin constant C is clearly a divergent series. In fact the summation formula here diverges for every x because of the supergeometric growth established for Bernoulli numbers. Euler, however, is not fazed: he has a plan for obtaining from such divergent series highly accurate approximations for both very large finite and infinite series.

Euler's idea is to add up the terms in the summation formula only "*until it begins to diverge*". For those unfamiliar with the theory of divergent series, this seems preposterous, but in fact it has sound theoretical underpinnings established much later; Euler's approach was ultimately fully vindicated by the modern theory of asymptotic series [12, 13, 15, 19]. Euler himself was probably confident of his results, despite the apparently shaky foundations in divergent series, because he was continually checking and rechecking his results by a variety of theoretical and computational methods, boosting his confidence in their correctness from many different angles. Let us see how Euler continues analyzing the sum of reciprocal squares, begun above.

First he recalls that for this particular function, he already knows the value of C by other means.

Above we demonstrated that the sum of the series to infinity is $= \frac{\pi\pi}{6}$, and therefore setting $x = \infty$, and $s = \frac{\pi\pi}{6}$, we have $C = \frac{\pi\pi}{6}$, because then all other terms vanish. Thus it follows that

$$1 + 1 - \frac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \mathfrak{E} - \text{etc.} = \frac{\pi\pi}{6}.$$

Next Euler pretends he doesn't already know the sum of the infinite series of reciprocal squares, and approximates it using his summation formula, thereby performing a cross-check on both methods.

149. If the sum of this series were not known, then one would need to determine the value of the constant C from another case, in which the sum were actually found. To this aim we set $x = 10$ and actually add up ten terms, obtaining⁷

$$\begin{array}{rcl}
 s & = & 1,549767731166540690 \quad . \\
 \text{Further, add } \frac{1}{x} & = & 0,1 \\
 \text{subtr. } \frac{1}{2xx} & = & 0,005 \\
 & & \hline
 & & 1,644767731166540690 \\
 \text{add } \frac{\mathfrak{A}}{x^3} & = & 0,000166666666666666 \\
 & & \hline
 & & 1,644934397833207356 \\
 \text{subtr. } \frac{\mathfrak{B}}{x^5} & = & 0,000000333333333333 \\
 & & \hline
 & & 1,644934064499874023 \\
 \text{add } \frac{\mathfrak{C}}{x^7} & = & 0,00000002380952381 \\
 & & \hline
 & & 1,644934066880826404 \\
 \text{subtr. } \frac{\mathfrak{D}}{x^9} & = & 0,000000000333333333 \\
 & & \hline
 & & 1,644934066847493071 \\
 \text{add } \frac{\mathfrak{E}}{x^{11}} & = & 0,0000000000757575 \\
 & & \hline
 & & 1,644934066848250646 \\
 \text{subtr. } \frac{\mathfrak{F}}{x^{13}} & = & 0,0000000000025311 \\
 & & \hline
 & & 1,644934066848225335 \\
 \text{add } \frac{\mathfrak{G}}{x^{15}} & = & 0,0000000000001166 \\
 \text{subtr. } \frac{\mathfrak{H}}{x^{17}} & = & \frac{71}{1,644934066848226430} = C.
 \end{array}$$

This number is likewise the value of the expression $\frac{\pi\pi}{6}$, as one can find by calculation from the known value of π . From this it is clear that, although the series \mathfrak{A} , \mathfrak{B} , \mathfrak{C} , etc. diverges, it nevertheless produces a true sum.

⁷Euler uses commas (as still done in Europe today) rather than points, for separating the integer and fractional parts of a decimal.

So, on the one hand the summation formula diverges for every x , and yet on the other it can apparently be used to make very close approximations, in fact arbitrarily close approximations, to C . How can this be?

Note that the terms Euler actually calculates appear to decrease rapidly, giving the initial appearance, albeit illusory, that the series converges. Examining the terms more closely, one can see evidence that their decrease is slowing in a geometric sense, which hints at the fact that the series actually diverges. Recall that Euler intends to sum only “*until it begins to diverge*”. How does he decide when this occurs? Notice that the series alternates in sign, and thus the partial sums bounce back and forth, apparently at first narrowing in, then broadening out as the terms themselves eventually increase due to rapid growth of the Bernoulli numbers. Euler knows to stop before the smallest bounce, with the expectation that the true sum he seeks always lies between any partial sum and the next one, and is thus bracketed most accurately if he stops just before the smallest term included.

Much later, through the course of the nineteenth century, mathematicians would wrestle with the validity, theory and usefulness of divergent series. Two (divergent) views reflected this struggle, and exemplify the evolution of mathematics:

“The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever. By using them, one may draw any conclusion he pleases and that is why these series have produced so many fallacies and so many paradoxes. ...I have become prodigiously attentive to all this, for with the exception of the geometrical series, there does not exist in all of mathematics a single infinite series the sum of which has been determined rigorously. In other words, the things which are most important in mathematics are also those which have the least foundation. ... That most of these things are correct in spite of that is extraordinarily surprising. I am trying to find a reason for this; it is an exceedingly interesting question.”, Niels Abel (1802–1829), 1826 [17, p. 973f].

“The series is divergent; therefore we may be able to do something with it”, Oliver Heaviside (1850–1925) [17, p. 1096].

Euler, long before this, was confident in proceeding according to his simple dictum “*until it begins to diverge*”. Indeed, it is astounding but true that the summation formula does behave exactly as Euler used it for many functions, including all the ones Euler was interested in. Today we know for

certain that such “asymptotic series” indeed do bracket the desired answer, and diverge more and more slowly for larger and larger values of x , thus are extremely useful for approximations [13, 15, 19][17, ch. 47].

One can explore the interplay of calculation versus accuracy achieved by different choices for x . A smaller choice for x will cause the summation formula to begin to diverge sooner, and with a larger final bounce, yielding less accuracy. While a larger x will ensure much more rapid achievement of a given level of accuracy, and greater bounding accuracy (as small as desired) for the answer, at the expense of having to compute a longer partial sum on the left hand side to get the calculation off the ground. Asymptotic series have become very important in applications of differential equations to physical problems [17, ch. 47].

Euler’s next application of the summation formula, in §150–153, is to approximate the sums of reciprocal odd powers. I remarked above that Euler’s very first application of the Bernoulli numbers was to determine the precise sums of all infinite series of reciprocal even powers in terms of Bernoulli numbers and π . Naturally he also would have loved to find similar formulas for the reciprocal odd powers, and he explores this at length using the summation formula. He produces highly accurate decimal approximations for sums of reciprocal odd powers all the way through the fifteenth, hoping to see a pattern analogous to the even powers, namely simple fractions times the relevant power of π . The first such converging series is the sum of reciprocal cubes $\sum_{i=1}^{\infty} 1/i^3$. Euler computes it accurately to seventeen decimal places. He is disappointed, however, to find that it is not near an obvious rational multiple of π^3 , nor does he have better luck with the other odd powers. Even today we know little about these sums of odd powers, although not for lack of trying.

Following this, in §154–156 Euler approximates π to seventeen decimal places using the inverse tangent and cotangent functions with the summation formula. He actually expresses his own amazement that one can approximate π so accurately with such an easy calculation!

8 How to determine (or not) factorials

I will showcase next Euler’s efficacious use of the summation formula to approximate finite sums of logarithms, and thus by exponentiating, to approximate very large factorials via the formula now known as Stirling’s asymptotic approximation. Notice particularly Euler’s ingenious determination of the Euler-Maclaurin constant in the summation formula, from Wallis’ infinite

product for π .

With the aid of a computational engine I will also briefly explore whether the summation formula can determine a factorial precisely, yielding surprising results.

To set the stage for Euler, notice that to estimate a factorial, one can estimate $\log(x!) = \log 1 + \log 2 + \dots + \log x$, using any base, provided one also knows how to find antilogarithms.

157. Now we want to use for z transcendental functions of x , and take $z = lx$ for summing hyperbolic⁸ logarithms, from which the ordinary can easily be recovered, so that

$$s = l1 + l2 + l3 + l4 + \dots + lx.$$

Because $z = lx$,

$$\int z dx = xlx - x,$$

since its differential is dxx . Then

$$\frac{dz}{dx} = \frac{1}{x}, \quad \frac{ddz}{dx^2} = -\frac{1}{x^2}, \quad \frac{d^3z}{1 \cdot 2dx^3} = \frac{1}{x^3}, \quad \frac{d^4z}{1 \cdot 2 \cdot 3dx^4} = -\frac{1}{x^4}, \quad \frac{d^5z}{1 \cdot 2 \cdot 3 \cdot 4dx^5} = \frac{1}{x^5}, \text{ etc.}$$

One concludes that

$$s = xlx - x + \frac{1}{2}lx + \frac{\mathfrak{A}}{1 \cdot 2x} - \frac{\mathfrak{B}}{3 \cdot 4x^3} + \frac{\mathfrak{C}}{5 \cdot 6x^5} - \frac{\mathfrak{D}}{7 \cdot 8x^7} + \text{etc.} + \text{Const.}$$

But for this constant one finds, when one sets $x = 1$, because then $s = l1 = 0$,

$$C = 1 - \frac{\mathfrak{A}}{1 \cdot 2} + \frac{\mathfrak{B}}{3 \cdot 4} - \frac{\mathfrak{C}}{5 \cdot 6} + \frac{\mathfrak{D}}{7 \cdot 8} - \text{etc.},$$

a series that, due to its great divergence, is quite unsuitable even for determining the approximate value of C .

158. Nevertheless we can not only approximate the correct value of C , but can obtain it exactly, by considering Wallis's expression for π provided in the *Introductio* [6, vol. 1, chap. 11]. This expression is

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot \text{etc.}}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot \text{etc.}}$$

⁸Think about why Euler calls this the "hyperbolic" logarithm.

Taking logarithms, one obtains from this

$$l\pi - l2 = 2l2 + 2l4 + 2l6 + 2l8 + 2l10 + l12 + \text{etc.}$$

$$-l1 - 2l3 - 2l5 - 2l7 - 2l9 - 2l11 - \text{etc.}$$

Setting $x = \infty$ in the assumed series, we have

$$l1 + l2 + l3 + l4 + \dots + lx = C + \left(x + \frac{1}{2}\right) lx - x,$$

thus $l1 + l2 + l3 + l4 + \dots + l2x = C + \left(2x + \frac{1}{2}\right) l2x - 2x$

and $l2 + l4 + l6 + l8 + \dots + l2x = C + \left(x + \frac{1}{2}\right) lx + xl2 - x,$

and therefore $l1 + l3 + l5 + l7 + \dots + l(2x - 1) = xlx + \left(x + \frac{1}{2}\right) l2 - x.$

Thus because

$$l\frac{\pi}{2} = 2l2 + 2l4 + 2l6 + \dots + 2l2x - l2x$$

$$- 2l1 - 2l3 - 2l5 - \dots - 2l(2x - 1),$$

letting $x = \infty$ yields

$$l\frac{\pi}{2} = 2C + (2x + 1) lx + 2xl2 - 2x - l2 - lx - 2xlx - (2x + 1) l2 + 2x,$$

and therefore

$$l\frac{\pi}{2} = 2C - 2l2, \text{ thus } 2C = l2\pi \text{ and } C = \frac{1}{2}l2\pi,$$

yielding the decimal fraction representation

$$C = 0,9189385332046727417803297,$$

thus simultaneously the sum of the series

$$1 - \frac{\mathfrak{A}}{1 \cdot 2} + \frac{\mathfrak{B}}{3 \cdot 4} - \frac{\mathfrak{C}}{5 \cdot 6} + \frac{\mathfrak{D}}{7 \cdot 8} - \frac{\mathfrak{E}}{9 \cdot 10} + \text{etc.} = \frac{1}{2}l2\pi.$$

159. Since we now know the constant $C = \frac{1}{2}l2\pi$, one can exhibit the sum of any number of logarithms from the series $l1 + l2 + l3 + \text{etc.}$ If one sets

$$s = l1 + l2 + l3 + l4 + \dots + lx,$$

then

$$s = \frac{1}{2}l2\pi + \left(x + \frac{1}{2}\right) lx - x + \frac{\mathfrak{A}}{1 \cdot 2x} - \frac{\mathfrak{B}}{3 \cdot 4x^3} + \frac{\mathfrak{C}}{5 \cdot 6x^5} - \frac{\mathfrak{D}}{7 \cdot 8x^7} + \text{etc.}$$

if the proposed logarithms are hyperbolic; if however the proposed logarithms are common, then one must take common logarithms also in the terms $\frac{1}{2}l2\pi + (x + \frac{1}{2})lx$ for $l2\pi$ and lx , and multiply the remaining terms

$$-x + \frac{\mathfrak{A}}{1 \cdot 2x} - \frac{\mathfrak{B}}{3 \cdot 4x^3} + \text{etc.}$$

of the series by 0,434294481903251827 = n . In this case the common logarithms are

$$\begin{aligned} l\pi &= 0,497149872694133854351268 \\ l2 &= 0,301029995663981195213738 \\ l2\pi &= 0,798179868358115049565006 \\ \frac{1}{2}l2\pi &= 0,399089934179057524782503. \end{aligned}$$

Example.

Find the sum of the first thousand common logarithms

$$s = l1 + l2 + l3 + \dots + l1000.$$

So $x = 1000$, and

$$\begin{aligned} lx &= 3,000000000000, \\ \text{and thus } xlx &= 3000,000000000000 \\ \frac{1}{2}lx &= 1,50000000000000 \\ \frac{1}{2}l2\pi &= 0,3990899341790 \\ &\quad \underline{3001,8990899341790} \\ \text{subtr. } nx &= 434,2944819032518 \\ &\quad \underline{2567,6046080309272}. \end{aligned}$$

Then

$$\begin{aligned} \frac{n\mathfrak{A}}{1 \cdot 2x} &= 0,0000361912068 \\ \text{subtr. } \frac{n\mathfrak{B}}{3 \cdot 4x^3} &= 0,0000000000012 \\ &\quad \underline{0,0000361912056} \\ \text{add } 2567,6046080309272 & \\ \text{the sum sought } s &= \underline{2567,6046442221328}. \end{aligned}$$

Now because s is the logarithm of a product of numbers

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots 1000,$$

it is clear that this product, if one actually multiplies it out, consists of 2568 figures, beginning with the figures 4023872, with 2561 subsequent figures.

One wonders how accurate such factorial approximations from the summation formula can actually be. Exponentiating Euler's summation formula above for a sum of logarithms produces the Stirling asymptotic approximation:

$$x! \approx \frac{\sqrt{2\pi x} x^x}{e^x} e^{\left(\frac{1}{1 \cdot 2x} - \frac{1}{3 \cdot 4x^3} + \frac{1}{5 \cdot 6x^5} - \frac{1}{7 \cdot 8x^7} + \cdots\right)}.$$

Since the summation formula diverges for each x , the accuracy of this approximation is theoretically limited. Yet the value sought always lies between those of successive partial sums. Moreover, from the asymptotic growth rate of Bernoulli numbers obtained earlier, approximately the first πx terms in the exponent might be expected to decrease (recall that for $x = 1000$ Euler used only two terms).

To explore the accuracy achievable with this formula, let us denote by $S(x, m)$ the approximation to $x!$ using the first m terms in the exponent. Beginning modestly with $x = 10$, calculations with Maple show that 3628800 is the only integer between $S(10, 2)$ and $S(10, 3)$, thus determining $10!$ on the nose. So although the summation formula has limited accuracy, it suffices to determine easily the integer $10!$ uniquely.

And for $x = 50$, one finds that 30414093201713378043612608166064768844377641568960512000000000000 is the only integer between $S(50, 26)$ and $S(50, 27)$, thus producing all 65 digits of $50!$. This striking accuracy leads us to ask:

Question: Can one obtain the exact value of any factorial this way?

There is an interplay here as x grows. Certainly the exponent becomes more accurately known for larger x , using a given number of terms, and in fact more precisely known from the diverging series generally, which improves for around $x\pi$ terms. On the other hand, it is then being exponentiated, and finally, multiplied times something growing, to produce the factorial approximation. So it is not so clear whether the factorial itself will always be sufficiently trapped to determine its integer value.

Continuing experimentally, let us compare with $x = 100$. First, note that $100!$ is approximately 9.33×10^{157} . Using the same number of terms, 27, as was needed above to determine uniquely all 65 digits of $50!$, one finds

that $S(100, 27)$ agrees with $100!$ for the first 82 digits. Thus it is giving more digits than when $x = 50$, but does not yet determine all the digits of $100!$. Further calculation shows that $100!$ is indeed the unique integer first bracketed by $S(100, 74)$ and $S(100, 75)$. In fact, one expects further improvement for 100π terms. While one still seems to have lots of terms to spare, one worries that, as x increases, with the number of decreasing terms in the summation only increasing linearly, i.e., as πx , the number of terms needed to bracket the factorial uniquely may be growing faster than this. In particular, when one doubled x from 50 to 100, the number of terms needed to determine the factorial increased from 27 to 75, more than doubling.

Both my theoretical analysis and further Maple computations ultimately confirm this fear, eventually answering the question in the negative. But the size of the factorials which are actually uniquely determined as integers by Euler's summation formula before it cannot keep up with all the digits is staggering, and includes every one of the 2568 digits that Euler determined $1000!$ has. I will provide a closer analysis and resolution of this phenomenon in an extended version of this paper.

9 Large binomials

In our last excerpt, Euler applies the summation formula to estimate the size of large binomial coefficients. I translate just one of his methods here, in which he coalesces two summation series term by term. As a sample application, Euler approximates the ratio $\binom{100}{50}/2^{100}$, despite the huge size of its parts, thus closely approximating the probability that if one tosses 100 coins, exactly equal numbers will land heads and tails.

160. By means of this summation of logarithms, one can approximate the product of any number of factors, that progress in the order of the natural numbers. This can be especially helpful for the problem of finding the middle or largest coefficient of any power in the binomial $(a + b)^m$, where one notes that, when m is an odd number, one always has two equal middle coefficients, which taken together produce the middle coefficient of the next even power. Thus since the largest coefficient of any even power is twice as large as the middle coefficient of the immediately preceding odd power, it suffices to determine the middle largest coefficient of an even power. Thus we have $m = 2n$ with middle coefficient expressed as

$$\frac{2n(2n-1)(2n-2)(2n-3)\cdots(n+1)}{1\cdot 2\cdot 3\cdot 4\cdots n}.$$

Setting this = u , one has

$$u = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots 2n}{(1 \cdot 2 \cdot 3 \cdot 4 \cdots n)^2},$$

and taking logarithms

$$lu = l1 + l2 + l3 + l4 + l5 + \cdots + l2n \\ - 2l1 - 2l2 - 2l3 - 2l4 - 2l5 - \cdots - 2ln.$$

161. The sum of hyperbolic logarithms is

$$l1 + l2 + l3 + l4 + \cdots + l2n = \frac{1}{2}l2\pi + \left(2n + \frac{1}{2}\right)ln + \left(2n + \frac{1}{2}\right)l2 - 2n \\ + \frac{\mathfrak{A}}{1 \cdot 2 \cdot 2n} - \frac{\mathfrak{B}}{3 \cdot 4 \cdot 2^3n^3} + \frac{\mathfrak{C}}{5 \cdot 6 \cdot 2^5n^5} - \text{etc.}$$

and

$$2l1 + 2l2 + 2l3 + 2l4 + \cdots + 2ln \\ = l2\pi + (2n + 1)ln - 2n + \frac{2\mathfrak{A}}{1 \cdot 2n} - \frac{2\mathfrak{B}}{3 \cdot 4n^3} + \frac{2\mathfrak{C}}{5 \cdot 6n^5} - \text{etc.}$$

Subtracting this expression from the former yields

$$lu = -\frac{1}{2}l\pi - \frac{1}{2}ln + 2nl2 + \frac{\mathfrak{A}}{1 \cdot 2 \cdot 2n} - \frac{\mathfrak{B}}{3 \cdot 4 \cdot 2^3n^3} + \frac{\mathfrak{C}}{5 \cdot 6 \cdot 2^5n^5} - \text{etc.} \\ - \frac{2\mathfrak{A}}{1 \cdot 2n} + \frac{2\mathfrak{B}}{3 \cdot 4n^3} - \frac{2\mathfrak{C}}{5 \cdot 6n^5} + \text{etc.},$$

and collecting terms in pairs

$$lu = l \frac{2^{2n}}{\sqrt{n\pi}} - \frac{3\mathfrak{A}}{1 \cdot 2 \cdot 2n} + \frac{15\mathfrak{B}}{3 \cdot 4 \cdot 2^3n^3} - \frac{63\mathfrak{C}}{5 \cdot 6 \cdot 2^5n^5} + \frac{255\mathfrak{D}}{7 \cdot 8 \cdot 2^7n^7} - \text{etc.}$$

...

162. ...

Second Example

Find the ratio of the middle term of the binomial $(1 + 1)^{100}$ to the sum 2^{100} of all the terms.

For this we wish to use the formula we found first,

$$lu = l \frac{2^{2n}}{\sqrt{n\pi}} - \frac{3\mathfrak{A}}{1 \cdot 2 \cdot 2n} + \frac{15\mathfrak{B}}{3 \cdot 4 \cdot 2^3 n^3} - \frac{63\mathfrak{C}}{5 \cdot 6 \cdot 2^5 n^5} + \text{etc.},$$

from which, setting $2n = m$, in order to obtain the power $(1 + 1)^m$, and after substituting the values of the letters \mathfrak{A} , \mathfrak{B} , \mathfrak{C} , \mathfrak{D} etc., one has

$$lu = l \frac{2^m}{\sqrt{\frac{1}{2}m\pi}} - \frac{1}{4m} + \frac{1}{24m^3} - \frac{1}{20m^5} + \frac{17}{112m^7} - \frac{31}{36m^9} + \frac{691}{88m^{11}} - \text{etc.}$$

Since the logarithms here are hyperbolic, one multiplies by

$$k = 0,434294481903251,$$

in order to change to tables, yielding

$$lu = l \frac{2^m}{\sqrt{\frac{1}{2}m\pi}} - \frac{k}{4m} + \frac{k}{24m^3} - \frac{k}{20m^5} + \frac{17k}{112m^7} - \frac{31k}{36m^9} + \text{etc.},$$

Now since u is the middle coefficient, the ratio sought is $2^m : u$, and

$$l \frac{2^m}{u} = l \sqrt{\frac{1}{2}m\pi} + \frac{k}{4m} - \frac{k}{24m^3} + \frac{k}{20m^5} - \frac{17k}{112m^7} + \frac{31k}{36m^9} - \frac{691k}{88m^{11}} + \text{etc.}$$

Now, since the exponent $m = 100$,

$$\frac{k}{m} = 0,0043429448, \quad \frac{k}{m^3} = 0,0000004343, \quad \frac{k}{m^5} = 0,0000000000,$$

yielding

$$\frac{k}{4m} = 0,0010857362$$

$$\frac{k}{24m^3} = 0,0000000181$$

$$\underline{0,0010857181}.$$

$$\text{Further } l\pi = 0,4971498726$$

$$l \frac{1}{2}m = \underline{1,6989700043}$$

$$l \frac{1}{2}m\pi = \underline{2,1961198769}$$

$$l \sqrt{\frac{1}{2}m\pi} = 1,0980599384$$

$$\frac{k}{4m} - \frac{k}{24m^3} + \text{etc.} = \underline{0,0010857181}$$

$$1,0991456565 = l \frac{2^{100}}{u}.$$

Thus $\frac{2^{100}}{u} = 12,56451$, and the middle term in the expanded power $(1 + 1)^m$ is to the sum of all the terms 2^{100} as 1 is to 12,56451.

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