Are All Infinities Created Equal?

Guram Bezhanishvili*†

Georg Ferdinand Ludwig Philip Cantor (1845–1918), the founder of set theory, and considered by many as one of the most original minds in the history of mathematics, was born in St. Petersburg, Russia in 1845. His parents, who were of Jewish descent, moved the family to Frankfurt, Germany in 1856. Georg entered the Wiesbaden Gymnasium at the age of 15, and two years later began his university career at Zürich. In 1863 he moved to the University of Berlin, which during Cantor's time was considered the world's center of mathematical research. Four years later Cantor received his doctorate from the great Karl Weierstrass (1815–1897). In 1869 Cantor obtained an unpaid lecturing post, which ten years later flourished into a full professorship, at the minor University of Halle. However, he never achieved his dream of holding a Chair of Mathematics at Berlin. It is believed that one of the main reasons for this was the rejection of his theories of infinite sets by the leading mathematicians of that time, most noticeably by Leopold Kronecker (1823–1891), a professor at the University of Berlin and a very influential figure in German mathematics, both mathematically and politically.

Cantor married in 1874 and had two sons and four daughters. Ten years later Georg suffered the first of the mental breakdowns that were to plague him for the rest of his life. He died in 1918 in a mental hospital at Halle. By that time his revolutionary ideas were becoming accepted by some of the leading figures of the new century. For example, one of the greatest mathematicians of the twentieth century, David Hilbert (1862–1943), described Cantor's new mathematics as "the most astonishing product of mathematical thought" [5, p. 359], and claimed that "no one shall ever expel us from the paradise which Cantor has created for us" [5, p. 353].

In this project we will learn about Cantor's treatment of infinite sets. We will discuss the cardinality of a set, the notion of equivalence of two sets, and study how to compare infinite sets with each other. We will introduce countable sets and show that many sets are countable, including the set of integers and the set of rational numbers. We will also discuss Cantor's diagonalization method which allows us to show that not every infinite set is countable. In particular, we will show that the set of real numbers is not countable. We will also examine the cardinal number $\aleph_0$, the first in the hierarchy of transfinite cardinal numbers, and obtain a method that allows us to create infinitely many transfinite cardinal numbers.

We will learn much of this by studying and working with the historical source [3], which is an English translation of two papers by Cantor [1, 2] that appeared in 1895 and 1897. More on Georg Cantor can be found in [4, 5, 6] and in the literature cited therein.

We begin by reading Cantor's definition of the cardinal number of a given set. Note that in this translation Jourdain uses "aggregate" instead of the more familiar "set."

1. Read carefully the following quote from Cantor.

> Every aggregate M has a definite "power," which we also call its "cardinal number."

*Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003; gbezhani@nmsu.edu.

†With thanks to Joel Lucero-Bryan.

We will call by the name "power" or "cardinal number" of M the general concept which, by means of our active faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements m and of the order in which they are given.

We denote the result of this double act of abstraction, the cardinal number or power of M, by

$$\overline{\overline{M}}.$$

What do you think Cantor means by "cardinal number"? Why? Given a set M consisting of ten round marbles, each of a different color, what is $\overline{\overline{M}}$?

2. Read the following quote from Cantor.

We say that two aggregates M and N are "equivalent," in signs

$$M \sim N \quad \text{or} \quad N \sim M,$$

if it is possible to put them, by some law, in such a relation to one another that to every element of each one of them corresponds one and only one element of the other.

In modern terminology describe what it means for two sets to be equivalent.

3. Prove the following claim of Cantor.

Every aggregate is equivalent to itself:

$$M \sim M.$$

4. Prove the following claim of Cantor.

If two aggregates are equivalent to a third, they are equivalent to one another, that is to say:

$$\text{from} \quad M \sim P \quad \text{and} \quad N \sim P \quad \text{follows} \quad M \sim N.$$

5. Read carefully the following quote from Cantor.

Of fundamental importance is the theorem that two aggregates M and N have the same cardinal number if, and only if, they are equivalent: thus,

$$\text{from } M \sim N, \text{ we get } \overline{\overline{M}} = \overline{\overline{N}},$$

and

$$\text{from } \overline{\overline{M}} = \overline{\overline{N}}, \text{ we get } M \sim N.$$

Thus the equivalence of aggregates forms the necessary and sufficient condition for the equality of their cardinal numbers.

Explain in your own words what Cantor means in the above.

6. Let $\mathbf{P}$ be the set of all perfect squares

$$\{0, 1,\ 4,\ 9,\ 16,\ 25,\ \ldots\},$$

and let $\mathbf{N}$ denote the set of all natural numbers

$$\{0, 1,\ 2,\ 3,\ 4,\ 5,\ \ldots\}.$$

From Cantor's statement above, do $\mathbf{P}$ and $\mathbf{N}$ have the same cardinality? Justify your answer.

7. Let $\mathbf{Z}$ denote the set of all integers. Do $\mathbf{N}$ and $\mathbf{Z}$ have the same cardinality? Justify your answer.

8. Let $\mathbf{N} \times \mathbf{N}$ denote the Cartesian product of $\mathbf{N}$ with itself; that is

$$\mathbf{N} \times \mathbf{N} = \{(n, m) : n, m \in \mathbf{N}\}.$$

Do $\mathbf{N}$ and $\mathbf{N} \times \mathbf{N}$ have the same cardinality? Justify your answer. Hint: Draw a picture of $\mathbf{N} \times \mathbf{N}$. Can you label each element of $\mathbf{N} \times \mathbf{N}$ by a unique natural number?

9. Let $\mathbf{Q}$ denote the set of all rational numbers; that is

$$\mathbf{Q} = \{\frac{a}{b} : a \in \mathbf{Z} \text{ and } b \in \mathbf{N} - \{0\}\}.$$

What is the cardinality of $\mathbf{Q}$? Justify your answer. Hint: Establish a 1-1 correspondence between $\mathbf{Q}$ and (a subset of) $\mathbf{Z} \times (\mathbf{N} - \{0\})$ and modify your solution to (8).

10. Read carefully the following quote from Cantor.

> If for two aggregates M and N with the cardinal numbers $\mathfrak{a} = \overline{\overline{M}}$ and $\mathfrak{b} = \overline{\overline{N}}$, both the conditions:
>
> (a) There is no part[1] of M which is equivalent to N,
>
> (b) There is a part N_1 of N, such that $N_1 \sim M$,
>
> are fulfilled, it is obvious that these conditions still hold if in them M and N are replaced by two equivalent aggregates M' and N'. Thus they express a definite relation of the cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$ to one another.
>
> Further, the equivalence of M and N, and thus the equality of $\mathfrak{a}$ and $\mathfrak{b}$, is excluded; for if we had $M \sim N$, we would have, because $N_1 \sim M$, the equivalence $N_1 \sim N$, and then, because $M \sim N$, there would exist a part M_1 of M such that $M_1 \sim M$, and therefore we should have $M_1 \sim N$; and this contradicts the condition (a).
>
> Thirdly, the relation of $\mathfrak{a}$ to $\mathfrak{b}$ is such that it makes impossible the same relation of $\mathfrak{b}$ to $\mathfrak{a}$; for if in (a) and (b) the parts played by M and N are interchanged, two conditions arise which are contradictory to the former ones.
>
> We express the relation of $\mathfrak{a}$ to $\mathfrak{b}$ characterized by (a) and (b) by saying: $\mathfrak{a}$ is "less" than $\mathfrak{b}$ or $\mathfrak{b}$ is "greater" than $\mathfrak{a}$; in signs
>
> $$\mathfrak{a} < \mathfrak{b} \text{ or } \mathfrak{b} > \mathfrak{a}.$$

[1] The modern terminology is "subset".

Describe in modern terminology when two cardinals $\mathfrak{a} = \overline{\overline{M}}$ and $\mathfrak{b} = \overline{\overline{N}}$ are in the relation $\mathfrak{a} < \mathfrak{b}$.

11. Prove the following claim of Cantor.

> We can easily prove that,
>
> $$\text{if } \mathfrak{a} < \mathfrak{b} \text{ and } \mathfrak{b} < \mathfrak{c}, \text{ then we always have } \mathfrak{a} < \mathfrak{c}.$$

12. Read carefully the following quote from Cantor.

> Aggregates with finite cardinal numbers are called "finite aggregates," all others we will call "transfinite aggregates" and their cardinal numbers "transfinite cardinal numbers."
>
> The first example of a transfinite aggregate is given by the totality of finite cardinal numbers ν; we call its cardinal number "Aleph-zero," and denote it by $\aleph_0$;

In the modern terminology, a set whose cardinal number is $\aleph_0$ is called "countable." What symbol is used today to denote the "totality of finite cardinal numbers ν"?

13. Prove the following claim of Cantor.

> The number $\aleph_0$ is greater than any finite number μ:
>
> $$\aleph_0 > \mu.$$

14. Prove the following claim of Cantor.

> On the other hand, $\aleph_0$ is the least transfinite cardinal number. If $\mathfrak{a}$ is any transfinite cardinal number different from $\aleph_0$, then
>
> $$\aleph_0 < \mathfrak{a}.$$

Hint: Let $\mathfrak{a} = \overline{\overline{A}}$. Can you define a 1-1 map from $\mathbf{N}$ into A? What can you deduce from this?

15. Let $[0,1]$ denote the set of all real numbers between 0 and 1. Show that $\aleph_0 < \overline{\overline{[0,1]}}$. We outline what is now known as Cantor's diagonalization method as one way to prove this. Represent real numbers in $[0,1]$ as infinite decimals (which do not end in infinitely repeating 9's). Assume that $\mathbf{N} \sim [0,1]$. Then to each infinite decimal one can assign a unique natural number, so the infinite decimals can be enumerated as follows:

$$\begin{gathered} .a_{11}a_{12}\ldots a_{1n}\ldots \\ .a_{21}a_{22}\ldots a_{2n}\ldots \\ \vdots \\ .a_{n1}a_{n2}\ldots a_{nn}\ldots \\ \vdots \end{gathered}$$

Can you construct an infinite decimal $.b_1b_2\ldots b_n\ldots$ such that $a_{nn} \neq b_n$ for each positive n? What can you conclude from this?

16. Let $\mathbf{R}$ denote the set of all real numbers. Is $\overline{\overline{\mathbf{R}}}$ strictly greater than $\aleph_0$? Justify your answer.

17. For a set M, let $\mathcal{P}(M)$ denote the set of all subsets of M; that is $\mathcal{P}(M) = \{N : N \subseteq M\}$. Prove the following claim of Cantor:

$$\overline{\overline{\mathcal{P}(M)}} > \overline{\overline{M}}.$$

Hint: Employ a generalized version of Cantor's diagonalization method. Assume that $M \sim \mathcal{P}(M)$. Then there is a 1-1 and onto function $f : M \to \mathcal{P}(M)$. Consider the set $N = \{m \in M : m \notin f(m)\}$. Can you deduce that $N \subseteq M$ is not in the range of f? Does this imply a contradiction?

18. Using the previous exercise, give an infinite increasing sequence of transfinite cardinal numbers.

Notes to the Instructor

This project is designed for an undergraduate course in discrete mathematics. It could be assigned as a three-week project on naive set-theory with an emphasis on 1-1 correspondences. Since some of Cantor's writings require nontrivial interpretations, it is advisable that in the beginning the instructor leads the class carefully, especially in reading Cantor's "definition" of cardinal number. The instructor may also wish to lead the class in discovering that the set of rational numbers is countable, and especially in using Cantor's diagonalization method to show that the set of real numbers is not countable. There is a shadow of the axiom of choice in Cantor's claim that $\aleph_0 < \mathfrak{a}$ for any transfinite cardinal number $\mathfrak{a}$ different from $\aleph_0$ (Exercise 14). The instructor may wish to spend a little bit of class time on giving an informal explanation of the main idea behind the axiom of choice.

References

[1] Cantor, G., *Beiträge zur Begründung der transfiniten Mengenlehre. I*, Mathematische Annalen **46** (1895), 481–512.

[2] Cantor, G., *Beiträge zur Begründung der transfiniten Mengenlehre. II*, Mathematische Annalen **49** (1897), 207–246.

[3] Cantor, G., *Contributions to the Founding of the Theory of Transfinite Numbers*, Philip Jourdain (translator), Dover Publications Inc., New York, 1952.

[4] Dunham, W., *Journey Through Genius. The Great Theorems of Mathematics*, John Wiley & Sons Inc., New York, 1990.

[5] Hollingdale, S., *Makers of Mathematics*, Penguin Books, New York, 1994.

[6] Laubenbacher, R., Pengelley, D., *Mathematical Expeditions: Chronicles by the Explorers*, Springer Verlag, New York, 1999.