

Excerpt from a letter of Monsieur Lamé to Monsieur Liouville on the question: Given a convex polygon, in how many ways can one partition it into triangles by means of diagonals?<sup>1</sup>

Translation copyright © 2004 by David Pengelley  
(Individual educational use only of this translation  
may be made without permission)

Extrait d'une lettre de M. Lamé à M. Liouville sur cette question: Un polygone convexe étant donné, de combien de manières peut-on le partager en triangles au moyen de diagonales?

*Journal de Mathématiques Pures et Appliquées (Journal de Liouville)*

**3** (1838), 505–507.

(<http://math-doc.ujf-grenoble.fr/JMPA/>)

(<http://gallica.bnf.fr/Catalogue/noticesInd/FRBNF34348784.htm>)

“The formula that you communicated to me yesterday is easily deduced from the comparison of two methods leading to the same goal.

“Indeed, with the help of two different methods, one can evaluate the number of decompositions of a polygon into triangles: by consideration of the sides, or of the vertices.

## I.

“Let  $ABCDEF\dots$  be a convex polygon of  $n + 1$  sides, and denote by the symbol  $P_k$  the total number of decompositions of a polygon of  $k$  sides into triangles. An arbitrary side  $AB$  of  $ABCDEF\dots$  serves as the base of a triangle, in each of the  $P_{n+1}$  decompositions of the polygon, and the triangle will have its vertex at  $C$ , or  $D$ , or  $F\dots$ ; to the triangle  $CBA$  there will correspond  $P_n$  different decompositions; to  $DBA$  another group of decompositions, represented by the product  $P_3P_{n-1}$ ; to  $EBA$  the group  $P_4P_{n-2}$ ; to  $FBA$ ,  $P_5P_{n-3}$ ; and so forth, until the triangle  $ZAB$ , which will belong to a final group  $P_n$ . Now, all these groups are completely distinct: their sum therefore gives  $P_{n+1}$ . Thus one has

$$(1) \quad P_{n+1} = P_n + P_3P_{n-1} + P_4P_{n-2} + P_5P_{n-3} + \dots + P_{n-3}P_5 + P_{n-2}P_4 + P_{n-1}P_3 + P_n.$$

## II.

“Let  $abcde\dots$  be a polygon of  $n$  sides. To each of the  $n - 3$  diagonals, which end at one of the vertices  $a$ , there will correspond a group of decompositions, for which this diagonal will serve as the side of two adjacent

---

<sup>1</sup>See a Memoir of Segner (*Novi Commentarii Acad. Petrop.*, vol. VII, p. 203). The author found equation (1) of M. Lamé; but formula (3) presents a much simpler solution. Formula (3) is no doubt due to Euler. It is pointed out without proof on page 14 of the volume cited above. The equivalence of equations (1) and (3) is not easy to establish. M. Terquem proposed this problem to me, achieving it with the help of some properties of factorials. I then communicated it to various geometers: none of them solved it; M. Lamé has been very successful: I am unaware of whether others before him have obtained such an elegant solution. J. LIOUVILLE

triangles: to the first diagonal  $ac$  corresponds the group  $P_3P_{n-1}$ ; to the second  $ad$  corresponds  $P_4P_{n-2}$ ; to the third  $ae$ ,  $P_5P_{n-3}$ , and so forth until the last  $ax$ , which will occur in the group  $P_3P_{n-1}$ . These groups are not totally different, because it is easy to see that some of the partial decompositions, belonging to one of them, is also found in the preceding ones. Moreover they do not include the partial decompositions of  $P_n$  in which none of the diagonals ending in  $a$  occurs.

“But if one does the same for each of the other vertices of the polygon, and combines all the sums of the groups of these vertices, by their total sum  $n(P_3P_{n-1} + P_4P_{n-2} + \cdots + P_{n-2}P_4 + P_{n-1}P_3)$  one will be certain to include all the partial decompositions of  $P_n$ ; each of these is itself repeated therein a certain number of times.

“Indeed, if one imagines an arbitrary such decomposition, it contains  $n-2$  triangles, having altogether  $3n-6$  sides; if one removes from this number the  $n$  sides of the polygon, and takes half of the remainder, which is  $n-3$ , one will have the number of diagonals appearing in the given decomposition. Now, it is clear that this partial decomposition is repeated, in the preceding total sum, as many times as these  $n-3$  diagonals have ends, that is  $2n-6$  times: since each end is a vertex of the polygon, and in evaluating the groups of this vertex, the diagonal furnished a group including the particular partial decomposition under consideration.

“Thus, since each of the partial decompositions of the total group  $P_n$  is repeated  $2n-6$  times in  $n(P_3P_{n-1} + P_4P_{n-2} + \cdots + P_{n-2}P_4 + P_{n-1}P_3)$ , one obtains  $P_n$  upon dividing this sum by  $2n-6$ . Therefore one has

$$(2) \quad P_n = \frac{n(P_3P_{n-1} + P_4P_{n-2} + \cdots + P_{n-2}P_4 + P_{n-1}P_3)}{2n-6}.$$

### III.

“The first formula (1) gives

$$P_3P_{n-1} + P_4P_{n-2} + \cdots + P_{n-2}P_4 + P_{n-1}P_3 = P_{n+1} - 2P_n,$$

and the second (2) gives

$$P_3P_{n-1} + P_4P_{n-2} + \cdots + P_{n-2}P_4 + P_{n-1}P_3 = \frac{2n-6}{n}P_n;$$

so finally

$$P_{n+1} - 2P_n = \frac{2n-6}{n}P_n,$$

or

$$(3) \quad P_{n+1} = \frac{4n-6}{n}P_n.$$

This is what was to be proven.”

Paris, 25 August, 1838