An Efficient Variable-Length Code Construction for Iterative Source-Channel Decoding

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Abstract—We present a novel variable-length code (VLC) construction which exhibits an inherent error correcting capability due to the exclusive presence of codewords with even Hamming weight. Besides error robustness, the proposed code construction features a similar codeword length distribution as Golomb-Rice codes, and therefore, in particular for sources with exponentially distributed symbols, has good source compression properties at the same time. We show that in a source channel coding framework with outer source encoding, inner channel encoding with a recursive convolutional code, and iterative decoding the proposed VLC construction can lead to significant performance improvements compared to fixed-length source encoding with optimized mappings. In particular, simulation results for the AWGN channel verify that for Gauss-Markov sources a performance close to the theoretical limit can be achieved.

Index Terms—Variable-length codes, joint source channel coding, iterative decoding, serially concatenated codes

I. INTRODUCTION

The robust transmission of variable-length encoded source signals over wireless channels has become an active research area during the last years. It is motivated by the increasing demand on multimedia and data services in third and fourth generation wireless networks. In such applications source compression is usually carried out using standardized techniques which, in order to achieve high compression gains, often employ variable-length codes (VLCs).

VLCs were originally designed for error-free transmission scenarios. In this form they are highly sensitive to transmission errors and suffer from error propagation; for example, single bit errors may lead to insertions and deletions of symbols, causing a loss of synchronization. In order to make VLCs applicable to the peculiarities of wireless channels, a variety of joint source-channel coding and decoding schemes addressing the error-resilient transmission of VLCs have been proposed in the literature. This includes the design of error-resilient VLCs, the joint optimization of VLCs and channel codes, the robust decoding of VLCs, and the joint decoding of VLCs and channel codes. For delay- and/or complexity-constrained transmission scenarios those combined techniques are often more advantageous than the classical separation of source and channel encoding.

A class of robust VLCs is given by reversible variable-length codes (RVLCs) [2]–[5], which are a well established technique for robust video compression. RVLCs exploit additional redundancy in the VLC code words in order to guarantee bidirectional decoding. If the number of error events is low, the impact of a synchronization loss can be mitigated. In [6] variable-length error-correcting codes (VLECs) were introduced, which combine the variable-length character of VLCs with distance constraints of channel codes at the cost of additional redundancy. VLECs are furthermore addressed in [7], [8] where efficient and improved algorithms for the code design are discussed.

A key component for a reliable transmission of VLCs is given by the VLC decoder. The first maximum a posteriori probability (MAP) decoder was introduced in [9], where an optimal VLC sequence estimation was carried out by the Viterbi algorithm, applied to an appropriate VLC trellis. Another trellis representation, which has become the standard trellis for VLC decoding, was introduced in [10]. It benefits from an efficient state representation and allows both a bitwise MAP and a posteriori probability (APP) decoding with the BCJR algorithm [11] and an optimal sequence estimation with the Viterbi algorithm. Later on, both exact and complexity-reduced approximate MAP decoding techniques for variable-length source codes were proposed by different authors, either by using a modified or list Viterbi algorithm [12]–[17] or by using sequential decoding approaches [18], [19].

Many approaches (e.g., [13]–[15]) exploit the redundancy due to the residual source correlation after source encoding for additional error protection. Generally, this allows for a less powerful forward error correction (FEC) code, resulting in a reduction of allocated bandwidth or latency for the overall transmission system. As reported in [20]–[24], a significant amount of residual source correlation, modeled as first-order Markov process, can be observed for several output parameters in state-of-the-art speech codecs and the MPEG audio compression scheme.

Also, recently, distributed source compression schemes based on Wyner-Ziv coding [25] have been proposed, e.g. for video signals (see, e.g., [26]–[28]). In these schemes, typically the majority of the encoding complexity is shifted to the decoder and therefore, they can be applied in situations where computational complexity is a critical resource at the encoder, for example in sensor networks. As a consequence, there exists a certain amount of residual source redundancy which cannot...
be removed due to the complexity constraints for the source encoder.

Further improvements can be achieved by an additional forward error correction combined with a joint decoding of source and channel code. Depending on the encoder structure this can be obtained by merging the source and the channel decoder, as proposed in [12], [29]–[32]. Since the concatenation of VLC and channel code essentially represents a serial code concatenation, another option is to apply iterative source-channel decoding (ISCD), which was proposed in [33] for VLCs and in [34]–[36] for fixed-length mappings. Further ISCD approaches for VLCs were presented by several authors [37]–[43] and differ in the realization of the employed soft-input/soft-output VLC decoder.

In the following, we present a novel error-robust variable-length code. The code word set features a length distribution similar to Golomb-Rice codes [44]. It solely contains code words with even Hamming weight (and thus constitutes an even-weight variable-length code, EW-VLC), which guarantees a minimum Hamming distance of \( d_{\text{min}} = 2 \) between all code sequences constructed from this code. As we will see, the proposed EW-VLC code features both error detection capabilities and good source compression properties at the same time, which turns out to be extremely useful in a joint source channel coding context with iterative decoding: the resulting source-channel encoding and decoding scheme outperforms the best known fixed-length mappings for ISCD [45] without introducing additional redundancy. The resulting encoding schemes are optimized both in terms of convergence properties and error-floor performance based on the design guidelines for serial concatenated codes given in [46]–[49] and an analysis of the extrinsic information transfer (EXIT) characteristics [50]. Simulation results are presented for a binary transmission over the AWGN channel.

This paper is organized as follows. Section II presents a brief overview of the transmission system and iterative source-channel decoding. In Section III we propose the new EW-VLC code construction and discuss some of its properties. Then, in Section IV the EW-VLC code is specifically tailored to iterative source channel decoding. Finally, simulation results are given in Section V.

II. TRANSMISSION SYSTEM

Consider the joint source-channel coding transmission scheme depicted in Fig. 1, which addresses the transmission of a source-channel encoded real-valued source \( \mathbf{U}^I \) over a white Gaussian noise (AWGN) channel. The proposed EW-VLC will be employed in the source encoder, and decoding of the source and the channel encoder is performed jointly by an iterative decoding algorithm.

A. Joint Source-Channel Encoder

We consider the case where redundancy due to both residual intra-frame correlation and non-uniform source symbol distribution is inherent in the source signal. Therefore, we assume that the source vector \( \mathbf{U} \) consists of \( K \) real-valued correlated source symbols \( U_k \in \mathbb{R} \). After \( W \)-bit quantization, we obtain the index vector \( \mathbf{I} \) with elements \( I_k \in \mathcal{I} \), \( \mathcal{I} = \{0, \ldots, 2^W - 1\} \), drawn from the distribution \( \Pr(I_k) \). The residual index correlation is modeled by a first-order Markov process, characterized by the index-transition probabilities \( \Pr(I_k|I_{k-1}) \). These probabilities are later exploited at the receiver side as additional a priori knowledge, leading to an enhanced error protection. In this context, they may be interpreted as a soft parity check. After quantization, the indices \( I_k \) are mapped symbol-wise to their binary representation, resulting in a length-\( N \) bit vector with elements \( B_n \in \{0, 1\} \). For comparison purposes we consider mappings realized by variable-length codes (VLCS) and in particular by the proposed EW-VLC with the codeword set \( C \) and the mean word length \( l \) as well as mappings with a fixed word length \( W \).

After source encoding, the binary source vector \( \mathbf{B} \) is permuted by an interleaver \( \pi \). The resulting bit vector \( \mathbf{B}' \) with elements \( B'_{\pi} \) is the input to the channel encoder, which consists of a recursive systematic convolutional (RSC) code with rate \( R_{\text{RSC}} = 1 \), punctured from a rate-1/2 mother code with generator polynomials \( (G_r, G_f) \), with feedback polynomial \( G_r \) and feed-forward polynomial \( G_f \). Puncturing is performed randomly [51], [52] such that a fraction \( p_{\text{sys}} \in [0, 1] \) of the systematic bits and a fraction \( p_{\text{par}} = 1 - p_{\text{sys}} \) of the parity bits are randomly removed from the bit stream. This puncturing method is beneficial since, for a given mother code, an optimization relies solely on the parameter \( p_{\text{sys}} \). Note that the puncturing pattern must be known at the decoder which can for instance be obtained by transmitting the seed of the random generator. Further properties of randomly punctured codes are discussed in [52].

B. Iterative Source-Channel Decoder

After transmission over a binary-input additive white Gaussian noise channel (BI-AWGN) with transmitted symbols \( X_m \in \{-1, 1\} \), \( E_s = E(|X|^2) = 1 \) and an additional white Gaussian i.i.d. noise sequence \( \mathbf{N} \) with the noise variance \( \sigma_N^2 = 1/(2E_s/N_0) \), the observation \( Y_m \) at the output of the channel are available to the decoder. They are characterized by the likelihood \( p_{X|Y}(y|x) \). Note that the realized transmitter structure is equivalent to a serial concatenation of two component codes which are separated by an interleaver [46]. Accordingly, joint decoding of the source and channel code can be realized by the iterative decoding scheme depicted in Fig. 1. We assume in this work that the transition probabilities are available perfectly at the source decoder, which can for example be obtained by transmitting them as side information. As an alternative, joint decoding and estimation of the source statistics is also possible, as in [53], [54]. In the following, we give a brief description of the iterative source-channel decoder discussed in [43].

The iterative decoder consists of two soft-input/soft-output (SISO) decoders, corresponding to the outer source encoder and the inner channel encoder. The decoders are realized by trellis-based a posteriori probability (APP) decoders. They are implemented in the logarithmic domain (Log-APP decoder)
and provide log-likelihood ratios (LLRs) [55] for the corresponding symbols at the encoder side. For channel decoding, we apply the BCJR algorithm [11] to the trellis representation of the underlying RSC code and assume perfect knowledge of the noise statistics $p_{Y|X}(y|x)$ at the decoder. Source decoding in the VLC case is carried out by the bit-based Log-APP VLC decoder proposed in [43]. By appropriately decoding in the VLC case is carried out by the bit-based

$$I_p$$

of the noise statistics we apply the BCJR algorithm [11] to the trellis representation and provide log-likelihood ratios (LLRs) [55] for the corre-

Fig. 1. Model of the transmission system containing source encoding, channel encoding, and iterative source-channel decoding for the AWGN channel.

and the maximum \( a \) posteriori estimator for estimating \( U_k \) and \( I_k \) are applicable in this case.

The iterative decoding process is based on the exchange of extrinsic information [55] between the two component decoders. Based on the observations \( Y = [Y_1, \ldots, Y_M] \) for the transmitted symbols \( X_m \) at the output of the communication channel and on the a priori LLRs \( A^{(c)} = [A_1^{(c)}, \ldots, A_N^{(c)}] \) for the interleaved symbols \( B'_n \), the inner channel decoder computes the vector \( L(B) \) of a posteriori LLRs \( L(B'_n | Y, A^{(c)}) \) for the interleaved source bits \( B'_n \). Similarly, given the vector of a priori LLRs \( A^{(s)} = [A_1^{(s)}, \ldots, A_N^{(s)}] \) the outer source decoder computes the vector \( L(B) \) of a posteriori LLRs \( L(B_n | A^{(s)}) \) for the bits \( B_n \) at the output of the source encoder. The extrinsic LLRs \( E_n^{(c)} \) and \( E_n^{(s)} \), which are exchanged between the two decoders, can now be obtained by subtracting the a priori LLRs \( A^{(c)} \) and \( A^{(s)} \) from the respective a posteriori LLRs \( L(B'_n | A^{(c)}) \) and \( L(B_n | A^{(s)}) \) as shown in Fig. 1.

After (de-)interleaving, the extrinsic LLRs provided by the one decoder become the a priori input of the other decoder. Since no extrinsic information is available at the output of the outer decoder during the initial decoding step, the a priori LLRs \( A_0^{(c)} \) of the inner decoder are initialized with zeros. The iterative process continues until no transmission errors can be observed anymore for the given block or an appropriate stopping criterion is satisfied.

C. Code Rate of the Transmission System

Since the source correlation ameliorates the achievable error protection, the overall code rate \( R \) of the transmission system relies on both the explicit redundancy of the channel code and the amount of residual source redundancy. Therefore, we associate the source redundancy with a code rate \( R_S \).

It can be separated into the product of the code rate \( R_{corr} \), which corresponds to the redundancy due to the intra-frame correlation, and the code rate \( R_{Map} \), which is associated with the redundancy introduced by the bit mapping or the VLC. With \( H(X) \) denoting the entropy of a random variable \( X \), the rate \( R_S \) can be expressed as

$$R_S = \frac{H(I_k) + (K - 1) \cdot H(I_k | I_{k-1})}{K \cdot H(I_k)} \cdot \frac{H(I_k)}{l} =: R_{corr}$$

Due to the serial concatenation of source and channel encoding, the overall code rate \( R \) of the transmission system is then given by the product of the code rate \( R_S \) of the source encoder and the code rate \( R_{RSC} \) of the RSC code:

$$R = R_S \cdot R_{RSC} = R_{corr} \cdot R_{Map} \cdot R_{RSC}.$$
1. Initialization of
   - the current code word length \( l = l_{\text{min}} \),
   - the number of assigned code words \( A = 0 \), and
   - the set of available length-\( l \) codewords \( \mathcal{Y}_l = \{ b | b \in \mathbb{B} \} \).

2. The code word candidates \( b \) available in the set \( \mathcal{Y}_l \) are separated into the set of code words with even Hamming weight,
   \[ \mathcal{E}_l := \{ b | b \in \mathcal{Y}_l, \; w_H(b) \text{ is even} \} \]
   and into the set of code words with odd Hamming weight,
   \[ \mathcal{O}_l := \{ b | b \in \mathcal{Y}_l, \; w_H(b) \text{ is odd} \} \]

3. We assign a number of \( \min \{ |\mathcal{E}_l|, |\mathcal{I}| - A \} \) code words \( b \in \mathcal{E}_l \) with even Hamming weight to the code word set \( \mathcal{C} \) of the EW-VLC.

4. The number of assigned code words, the set of available code words, and the current code word length are updated according to
   \[ A \leftarrow A + \min \{ |\mathcal{E}_l|, |\mathcal{I}| - A \} \]
   \[ \mathcal{Y}_{l+1} = \{ b = [b_1, \ldots, b_{l+1}] | [b_1, \ldots, b_l] \in \mathcal{O}_l \} \]
   \[ l \leftarrow l + 1 \]

5. If all code words are assigned (\( A = |\mathcal{I}| \)), the code construction is completed; otherwise return to step 2.

After constructing the code word set \( \mathcal{C} \), the codewords are assigned to the symbols \( I_k \) in \( \mathcal{I} \) such that highly probable symbols are represented by short codewords. An example for the EW-VLC is given in Table I for the English alphabet and a minimum code-word length \( l_{\text{min}} = 3 \). The probabilities \( \Pr(I_k) \) of the letters in Table I are taken from [5].

For the performance of other VLC construction algorithms for the English alphabet we refer to Tab. I in [5].

### Table I

EW-VLC with \( l_{\text{min}} = 3 \) for the English alphabet. The probabilities \( \Pr(I_k) \) for the letters \( I_k \) are taken from [5].

<table>
<thead>
<tr>
<th>( I_k )</th>
<th>( \Pr(I_k) )</th>
<th>( \text{EW-VLC} )</th>
<th>( I_k )</th>
<th>( \Pr(I_k) )</th>
<th>( \text{EW-VLC} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>0.148786</td>
<td>000</td>
<td>F</td>
<td>0.024553</td>
<td>010 001</td>
</tr>
<tr>
<td>T</td>
<td>0.093541</td>
<td>011</td>
<td>M</td>
<td>0.023619</td>
<td>100 001</td>
</tr>
<tr>
<td>A</td>
<td>0.088337</td>
<td>010</td>
<td>C</td>
<td>0.020817</td>
<td>111 001</td>
</tr>
<tr>
<td>O</td>
<td>0.072485</td>
<td>110</td>
<td>W</td>
<td>0.018682</td>
<td>001 000 1</td>
</tr>
<tr>
<td>R</td>
<td>0.068722</td>
<td>001 1</td>
<td>G</td>
<td>0.015212</td>
<td>010 000 1</td>
</tr>
<tr>
<td>N</td>
<td>0.064985</td>
<td>010 1</td>
<td>Y</td>
<td>0.015212</td>
<td>100 001</td>
</tr>
<tr>
<td>H</td>
<td>0.058313</td>
<td>100 1</td>
<td>B</td>
<td>0.012677</td>
<td>111 000 1</td>
</tr>
<tr>
<td>I</td>
<td>0.056445</td>
<td>111 1</td>
<td>V</td>
<td>0.011609</td>
<td>001 000 01</td>
</tr>
<tr>
<td>S</td>
<td>0.053578</td>
<td>001 01</td>
<td>K</td>
<td>0.008674</td>
<td>010 000 01</td>
</tr>
<tr>
<td>D</td>
<td>0.047708</td>
<td>010 01</td>
<td>X</td>
<td>0.001468</td>
<td>100 000 01</td>
</tr>
<tr>
<td>L</td>
<td>0.041233</td>
<td>100 01</td>
<td>Y</td>
<td>0.000801</td>
<td>111 000 01</td>
</tr>
<tr>
<td>U</td>
<td>0.027622</td>
<td>111 01</td>
<td>Q</td>
<td>0.000801</td>
<td>001 000 001</td>
</tr>
<tr>
<td>P</td>
<td>0.025754</td>
<td>001 001</td>
<td>Z</td>
<td>0.000534</td>
<td>010 000 001</td>
</tr>
</tbody>
</table>

### Proposition 1

The code in \( \mathcal{C} \) is a prefix code.

### Proof

Let \( C_l \) denote the set of VLC codewords with the maximal length \( l \). For \( l = l_{\text{min}} \), all codewords in \( C_{l_{\text{min}}} \) have the same lengths. For \( l \rightarrow l + 1 \), a codeword of length \( l + 1 \) in \( C_{l+1} \) has a prefix of length \( l \) in \( C_l \) by construction. Since \( \mathcal{E}_l \subseteq C_l \) but \( \mathcal{E}_l \cap \mathcal{O}_l = \emptyset \), \( C_{l_{\text{min}}} \) is prefix-free.

### Proposition 2

The code in \( C \) has a minimum Hamming distance of two.

### Proof

All codewords in \( C \) have even Hamming weight. Consider two codeword sequences \( B \) and \( B \) of the same bit length \( N \), which contain at least one different source symbol. Since \( B \) and \( B \) have even Hamming weight, they must differ in at least two bit positions, leading to \( d_{\text{min}} = 2 \).

The following theorem addresses the length distribution of EW-VLCs which has some important implications on the source compression performance of these codes.

### Theorem 1

Let the length distribution of a VLC be defined by the number \( A_l \) of length-\( l \) code words. Then, the length distribution \( A_l^{(GR)} \) of the code-word set \( C \) is equivalent to the length distribution \( A_l^{(GR)} \) of a Golomb-Rice code.

### Proof

By construction, the length distribution \( A_l^{(C)} \) of the code-word set \( C \) is given by
\[
A_l^{(C)} = \begin{cases} 
0 & \text{for } l \notin [l_{\text{min}}, l_{\text{max}}], \\
2^{l_{\text{min}} - 1} & \text{for } l_{\text{min}} \leq l < l_{\text{max}}, \\
|\mathcal{I}| - (l_{\text{max}} - l_{\text{min}}) \cdot 2^{l_{\text{min}} - 1} & \text{for } l = l_{\text{max}}.
\end{cases}
\]

The length distribution \( A_l^{(GR)} \) of a Golomb-Rice code with \( q = \lfloor i/2^k \rfloor \) for \( k \in \mathbb{N} \), which encodes an integer number \( i \) to a code word composed of a length-\( q \) prefix and a length-\( k \) suffix, is given by
\[
A_l^{(GR)} = \begin{cases} 
0 & \text{for } l \leq k, \\
2^k & \text{for } l > k.
\end{cases}
\]

By applying the Golomb-Rice code to the finite set \( \mathcal{I} = \{0, \ldots, 2^W - 1\} \), the equality of both length distributions becomes obvious for \( k = l_{\text{min}} - 1 \).

From Theorem 1 it follows that both the Golomb-Rice code and the EW-VLC lead to the same code rate \( R_{\text{Map}} \). The connection between the length distributions of the proposed EW-VLC and a Golomb-Rice code becomes furthermore obvious from Table I. Since Golomb-Rice codes are known to be nearly optimal for sources with exponentially distributed symbols, we can expect from Theorem 1 that the proposed EW-VLC shows a good performance in terms of compression and efficiency. Furthermore, due to the minimum Hamming distance of two the EW-VLC construction exhibits an error-correction capability. This property becomes particularly important if this code is employed in a joint source-channel coding framework as described in Section II.

### C. Performance

Table II verifies the good compression properties of the EW-VLC which we expect from Theorem 1. For the distribution of the English alphabet and for a Laplacian, a Gaussian, and a uniform distribution with \( |\mathcal{I}| = 32 \) the code rate \( R_{\text{Map}} \) of the EW-VLC is compared with the code rates achieved by a Huffman code and by asymmetric and symmetric reversible
variable-length codes (RVLCs) from [4], [5], respectively. It is evident from Table II that the EW-VLC outperforms the RVLCs with respect to their compression properties. Furthermore, as a comparison with Table III from [8] reveals, the performance of the EW-VLC is similar to the performance of the VLEC from [8] for the case \( d_{\text{min}} = 2 \). However, the VLC construction introduced in the previous section is much simpler than the iterative approach proposed in [8].

### Table II

<table>
<thead>
<tr>
<th></th>
<th>Engl. alphabet</th>
<th>Code rate ( R_{\text{Map}} )</th>
<th>Laplace</th>
<th>Gaussian</th>
<th>uniform</th>
</tr>
</thead>
<tbody>
<tr>
<td>Huffman</td>
<td>0.9916</td>
<td>0.9926</td>
<td>0.9899</td>
<td>0.991</td>
<td>0.991</td>
</tr>
<tr>
<td>asym. RVLC [4]</td>
<td>0.9726</td>
<td>0.9818</td>
<td>0.9736</td>
<td>0.9568</td>
<td>0.9091</td>
</tr>
<tr>
<td>asym. RVLC [5]</td>
<td>0.9466</td>
<td>0.9667</td>
<td>0.9484</td>
<td>0.9091</td>
<td>0.9091</td>
</tr>
<tr>
<td>sym. RVLC [4]</td>
<td>0.8640</td>
<td>0.9122</td>
<td>0.8944</td>
<td>0.7692</td>
<td>0.7692</td>
</tr>
<tr>
<td>sym. RVLC [5]</td>
<td>0.8307</td>
<td>0.8993</td>
<td>0.8852</td>
<td>0.7143</td>
<td>0.7143</td>
</tr>
</tbody>
</table>

In particular, for Laplacian and Gaussian distributions the performance of EW-VLCs is close to the one of Huffman codes. This observation confirms our conclusion from Theorem 1. Finally, comparing Table I with Table I from [5] and Table III from [8] reveals that EW-VLCs lead to a shorter maximum codeword length than asymmetric RVLCs from [4], [5], whereas the maximum codeword length is equivalent to the one for the VLECs from [8] with \( d_{\text{min}} = 2 \). This is of relevance since the complexity of symbol-based decoding algorithms (e.g., [33], [38], [40]) depends on the differences between maximum and minimum codeword length.

### IV. Encoder Design for Iterative Source-Channel Decoding

In this section we address an optimization of source and channel encoder in Fig. 1 for iterative source-channel decoding. In particular, the properties of established bit mappings and variable-length codes with respect to the design guidelines for serial concatenated channel codes will be analyzed. We will show in the following that both conventional quantizer bit mappings and VLCs lead to several drawbacks which can be avoided by EW-VLCs. In addition, for both EW-VLCs and a fixed-length source encoding scheme with the optimized bit mappings from [45] we present optimization results for the inner channel encoder.

#### A. Convergence Analysis and Design Criteria

A powerful framework for the analysis of the convergence behavior of the iterative decoder is given by EXIT characteristics [50]. Based on information theoretic measures this approach allows for precise performance predictions of concatenated codes under iterative decoding without the need to fully simulate the iterative decoder.

In order to compute EXIT characteristics SISO decoders are treated as information filters. They are characterized by the amount of mutual information \( I_{\text{extr}}^{(s/c)} := I(E^{(s/c)}; B) \) between their extrinsic output \( E^{(s/c)} \) and the corresponding quantity \( B \) at the encoder, given the corresponding mutual information \( I_{\text{apri}}^{(s/c)} := I(A^{(s/c)}; \mathcal{B}) \) between the \( a \) priori input \( A^{(s/c)} \) and the underlying symbols \( B \). With the mutual information \( I_{\text{c}} := I(X_m; Y_m) \) between the received and transmitted symbols \( X_m \) and \( Y_m \) at the input and output of the channel, resp., the EXIT characteristics specifying the source and the channel decoder are defined by the mappings

\[
\chi^{(s)}: I_{\text{apri}}^{(s)} \mapsto I_{\text{extr}}^{(s)} = \chi^{(s)}(I_{\text{apri}}^{(s)}) \quad \text{and} \quad \chi^{(c)}: I_{\text{apri}}^{(c)} \mapsto I_{\text{extr}}^{(c)} = \chi^{(c)}(I_{\text{apri}}^{(c)} I_{\text{c}}).
\]

Note that, since the inner decoder also utilizes the \( a \) priori input for the channel observations, the EXIT characteristic of the inner channel decoder is conditioned on \( I_{\text{c}} \).

In order to avoid a convergence towards high bit error rates, the components of the encoder should be chosen in such a way that the EXIT characteristic \( \chi^{(c)} \) lies above \( \chi^{(s/c)} \) and intersections at \( (I_{\text{apri}}^{(s/c)}, I_{\text{extr}}^{(s/c)}) < (1, 1) \) bits/channel use\(^2\) are avoided. As a consequence, the inner encoder must be recursive [49], [50], and the minimum Hamming distance \( D_{\text{min}} \) of the outer encoder must satisfy \( D_{\text{min}} \geq 2 \). Both recommendations confirm the design guidelines derived from the analysis of the upper bound to the bit error probability in [46].

Another design criterion follows from the area property of EXIT characteristics, shown in [56] for the binary erasure channel (BEC). It states that the area \( \mathcal{A}(\chi^{(s/c)}) \) under the inverse EXIT characteristic of the outer decoder corresponds to its code rate \( R^{(o)} \), while the area \( \mathcal{A}(\chi^{(c)}) \) under the inner EXIT characteristic \( \chi^{(c)} \) is lower or equal to the ratio of the mutual information \( I_{\text{c}} \) of the communication channel and the inner code rate \( R^{(i)} \):

\[
\mathcal{A}(\chi^{(s/c)}) = R^{(o)} \quad \text{and} \quad \mathcal{A}(\chi^{(c)}) \leq I_{\text{c}} / R^{(i)}.
\]

Since equality holds for \( R^{(i)} = 1 \) it is reasonable to choose inner rate-1 codes. Otherwise, the transmission system may suffer from a capacity loss [56] which may increase the decoding threshold towards higher channel signal-to-noise ratios (SNRs).

#### B. Inverse EXIT Characteristics for the Inner Source Decoder

For both standard and optimized bit mappings, different types of RVLCs, and the EW-VLC introduced in Section III inverse EXIT characteristics for the inner source decoder [43] are depicted in Fig. 2(a). Here, the source signal is modeled by a first-order autoregressive process which is realized by filtering uncorrelated Gaussian noise with a recursive filter \( G(z) = z/(z - a) \) where \( a \) is the correlation coefficient. The EXIT characteristics are obtained for source vectors of \( K = 20000 \) symbols, a correlation coefficient \( a = 0.9 \), and a 4-bit scalar quantizer with a Lloyd-Max quantizer.

As shown in Fig. 2(a), for standard bit mappings as the Gray mapping or the natural binary mapping, the outer source decoder provides only a low amount of information

\(^2\)Note that the maximum amount of information which can be obtained on a bit \( B \) from a noisy observation \( Y \) is given as \( I(B; Y) = H(B) - H(B|Y) \leq H(B) \). Throughout this paper we assume that by appropriate source coding \( H(B) = 1 \) can be achieved which is fulfilled in good approximation, e.g., for all VLCs with \( d_f \geq 2 \) in Fig. 2(a).
at its extrinsic output, given perfect a priori information ($I_{apri}^{(s)} = 1$ bits/channel use). Early intersections with the EXIT function of the inner decoder are the consequence, leading to a poor performance of the transmission system.

To overcome this problem, optimized mappings have been proposed in [45], [57], which are based on a maximization of the distortion for the case of one-bit errors and on a maximization of the extrinsic information for optimal a priori information, respectively. The inverse EXIT characteristic for the mapping from [45] is included in Fig. 2(a) which shows the superiority of the optimized mappings to the conventional ones. However, we are still not able to obtain iterative decoder convergence at $(I_{apri}^{(s)}, I_{extr}^{(s)}) = (0, 1)$ bits/channel use.

A similar behavior of the iterative decoder can be expected for the asymmetrical RVLC proposed in [4]. By considering an additional symmetry constraint in the RVLC design [4], the robustness can be further improved, however, at the cost of a decreased code rate $R_{map}$. In this case, the mean word length $\overline{l}$ is larger than the quantization word length $W$, i.e., the symmetrical RVLC introduces some redundancy.

Neither the analyzed mappings nor the RVLCs from [4] fulfill the condition $d_{min}^{(s)} \geq 2$. As a consequence, transmission systems which employ one of these codes can generally not benefit from an interleaver gain. This drawback can be avoided with the EW-VLC introduced in Section III. As Fig. 2(a) shows, $(I_{apri}^{(s)}, I_{extr}^{(s)}) = (1, 1)$ bits/channel use is achieved in the EXIT characteristics, and in addition a slight compression can be obtained compared to the bit mappings with constant word length. Assuming an appropriate (recursive) inner encoder, convergence at $(I_{apri}^{(s)}, I_{extr}^{(s)}) = (1, 1)$ bits/channel use becomes possible, and the interleaver gain can be exploited. On the other hand, RVLCs which satisfy the distance constraint $d_{min}^{(s)} \geq 2$ are proposed in [5], and their inverse EXIT characteristics are included in Fig. 2(a) as well. However, as we can see in Table II and conclude from the area under the inverse EXIT characteristic, the code rate is decreased compared to the EW-VLC; i.e., additional redundancy is introduced due to the additional reversibility constraint.

C. Code Matching for the Inner and Outer Encoder

In the following, we focus on the design of the inner RSC code. The goal is to find a good-matching inner code whose EXIT characteristic lies above the inverse EXIT characteristic of the outer source decoder. Therefore, we restrict ourselves to the EW-VLCs from the previous section, since they offer the best trade-off between amount of compression and distance properties. As a benchmarking scheme for our EW-VLC-based approach we consider fixed-length quantizer indices with an optimized bit mapping [45] in the source encoding stage.

As Fig. 2(a) shows, the outer source decoder provides only a small amount of extrinsic information $I_{extr}^{(s)}$ for $I_{apri}^{(s)} \leq 0.25$ bits/channel use for both EW-VLC and optimized bit mapping. Therefore, in order to avoid an early intersection of the EXIT characteristics, the inner channel encoder has to be chosen such that in absence of a priori knowledge ($I_{apri}^{(c)} = 0$ bits/channel use) a minimum amount of extrinsic information $I_{extr}^{(c)}(0) \geq 0.1$ bits/channel use is guaranteed.

For rate-1 RSC codes, which are obtained by puncturing the systematic bits, it was observed in [50] that only codes whose feed-forward polynomials consist of a single coefficient are able to generate extrinsic information $I_{extr}^{(c)}(0) > 0$ bits/channel use without additional a priori knowledge. However, especially for a short code memory, these codes have the drawback that only few candidates are available such that the code design becomes quite constrained.

As a remedy, we apply randomly punctured (RP) RSC codes [52] as suggested in Section II-A. Compared to the optimization of conventional puncturing patterns, the design of appropriate RP-RSC codes is drastically simplified. The puncturing process is parametrized by a single parameter, the puncturing rate $p_{sys}$. The result of the optimization for the EW-VLC from Section III is shown in Fig. 2(b) for the BIAWGN channel with $E_b/N_0 = 2$ dB and $E_b = E_s/R$. Here, the overall code rate $R = 0.66$ is considered, obtained for the EW-VLC and the source parameters from above. A good matching code is found by randomly puncturing a fraction $p_{sys} = 92\%$ of the systematic bits and a fraction $p_{par} = 8\%$.
of the parity bits from the memory-3 RSC code with generator polynomials \((13, 14)_8\).

For the optimized bit mapping from [45] we apply the memory-5 RSC code [58], where the corresponding EXIT characteristic is shown in Fig. 2(b) for the same parameters of the BI-AWGN channel. It is obtained by puncturing all systematic bits from the rate-1/2 mother code with the generator polynomials \((75, 1)_8\). Since in this case the EXIT characteristic for the source decoder does not achieve the point \((I_{\text{apri}}, I_{\text{extr}}) = (1, 1)\) bits/channel use, an RSC code with larger memory must be applied in order to lower the error floor.

V. SIMULATION RESULTS

In the following we verify the good performance of the EW-VLC from Section III in the context of iterative source-channel decoding. Monte Carlo simulations are performed for the BPSK-modulated BI-AWGN channel by using the code constructions from Section IV-C. For the sake of completeness, a second reference system based on the asymmetric RVLC (aRVLC) from [5] and the rate-1 randomly punctured RSC code \((13, 14)_8\) with \(p_{\text{sys}} = 92\%\) is considered as well.

The results are presented in Fig. 3 for the symbol error rate \((\text{SER}_L)\) based on the Levenshtein metric\(^3\) and for the distortion in terms of reconstruction signal-to-noise ratio \((R - SNR)\). In order to allow a fair comparison between different approaches the channels are parametrized by the channel SNR \(E_b/N_0\) related to the transmit energy per information bit \(E_b = E_s/R\). The simulations are carried out for source vectors of \(K = \{200, 2000, 20000\}\) symbols, a correlation coefficient \(a = 0.9\), and a 4-bit scalar quantization with a Lloyd-Max quantizer, similar to the parameters chosen in Section IV-B.

All coding schemes achieve approximately the same overall code rate of \(R \approx 0.66\). All interleavers are realized by \(s\)-random interleavers [59].

Fig. 3 shows the good performance of the EW-VLC combined with the randomly punctured RSC code \((13, 14)_8\). For a block length of \(K = 20000\) symbols and \(E_b/N_0 \geq 1.6\) dB no error could be observed within our simulation accuracy after 40 iterations\(^4\), and we achieve a threshold within 0.6 dB of the capacity limit for \(R = 0.66\). We outperform the coding scheme based on the aRVLC [5] by 0.2 dB. Compared to the fixed-length approach with an optimized mapping from [45] and a memory-5 RSC code a gain of 0.5 dB in channel SNR with respect to clear channel quality can be observed in Fig. 3(b). For the optimized mapping convergence is obtained after 10 iterations, even though the transmission suffers from a residual \(\text{SER}\) of approximately \(2 \cdot 10^{-4}\).

For an intermediate block length of \(K = 2000\) symbols the performances of both systems deteriorate. Despite the VLC approach now also shows an error-floor it still outperforms the optimized mapping: for \(E_b/N_0 \geq 1.75\) dB and \(K = 2000\) symbols the achieved distortion is comparable to the one obtained by the fixed-length-based scheme for a block length of \(K = 20000\) symbols. If the block length is further reduced to \(K = 200\) symbols the optimized mapping becomes superior for the channel SNR regime shown in Fig. 3.

The \(\text{SER}\) comparison in Fig. 3(a) for different block/interleaver lengths shows the behavior we expect according to the discussion from Section IV-B: due to a minimum distance of \(d_{\min}^{(o)} = 1\) the system based on the optimized mapping suffers from a residual error floor for \(E_b/N_0 \geq 2.25\) dB. In contrast, the EW-VLC with \(d_{\min}^{(o)} = 2\) takes advantage of the available interleaver gain: while the \(\text{SER}\) only slowly decreases for \(E_b/N_0 \geq 2.5\) dB and a block length of \(K = 2000\) symbols, no error event could be observed within our simulation accuracy for \(E_b/N_0 \geq 1.6\) dB and \(K = 20000\) symbols.

Finally, the special case of an uncorrelated source \((a = 0)\) is considered in Fig. 4 for a block length of \(K = 20000\) symbols. As we can see, the joint source-channel coding scheme employing the EW-VLC clearly outperforms the scheme based on the optimized mapping [45]. The large gains can be explained from the properties of the EXIT functions for the optimized mapping: for optimal \(a\) priori information \(I_{\text{apri}}^{(s)} = 1\) bits/channel use the source decoder employing the optimized mapping provides only the extrinsic information \(I_{\text{extr}}^{(s)} = 0.187\) bits/channel use, leading to early intersections of the EXIT functions. In contrast, the EXIT functions in the EW-VLC case achieve the point \((1, 1)\) bits/channel use in the EXIT chart. It is thus straightforward to find a matching channel code which leads to a good performance as shown in Fig. 4. Note that these results hold similarly for small correlation coefficients \(a > 0\) with the difference that the extrinsic information \(I_{\text{extr}}^{(s)}\) is slightly higher than for \(a = 0\).

VI. CONCLUSIONS

We have presented a simple variable-length code construction which is well suited for transmission systems employing iterative source-channel decoding. While providing good compression properties compared to, e.g., RVLCs the proposed EW-VLC guarantees a minimum Hamming distance of two between equal-length code sequences. Both features are the key to transmission systems which are extremely efficient in terms of required overhead for error protection and robust at the same time. As the simulation results for correlated Gauss-Markov sources show, the proposed EW-VLC facilitates reliable communication close to the AWGN channel capacity solely by exploiting residual source redundancy. Finally, a comparison shows that the proposed EW-VLC-based scheme outperforms the best fixed-length bit mapping optimized for iterative source-channel decoding. These results suggest to employ well-designed variable-length codes instead of fixed-length mappings in the context of iterative source-channel decoding.

REFERENCES


\(^3\)The Levenshtein metric of two sequences gives the number of insertions, deletions, and/or substitutions which are needed to transform one sequence into the other. An efficient approximation is proposed in [6].

\(^4\)Note that in this case a very low error floor exists which may not be measured by Monte Carlo simulations in a reasonable time.


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