4.3 Minimization problems ; Duality

Solving standard minimization problems using the dual

Standard Minimization problems that have only ≥ constraints can be transformed into maximization problems which are much easier to pivot and solve.

Example (Exercise 12 page 180):

Minimize \( w = 3y_1 + 2y_2 \)
Subject to: \( 2y_1 + 3y_2 \geq 60 \)
\( y_1 + 4y_2 \geq 50 \)
\( y_1, y_2 \geq 0 \)

Instead of minimizing, we will think of the problem as maximizing. Let’s go through the steps of building the first simplex matrix of the dual problem. We begin by writing down a matrix that represents the constraints and the objective function this matrix called augmented matrix.

\[
\begin{bmatrix}
2 & 3 & 60 \\
1 & 4 & 40 \\
3 & 2 & 0 \\
\end{bmatrix}
\]

We will take the TRANSPOSE of the coefficient part for our new matrix and swap the position of the RHS and objective function coefficients:

\[
\begin{bmatrix}
2 & 1 & 3 \\
3 & 4 & 2 \\
60 & 40 & 0 \\
\end{bmatrix}
\]

This matrix is called the transpose of the first matrix. From this second matrix, we create the following maximum problem (using x instead of y).

Max \( z = 60x_1 + 40x_2 \)
Subject to: \( 2x_1 + x_2 \leq 3 \)
\( 3x_1 + 4x_2 \leq 2 \)
\( x_1, x_2 \geq 0 \)
Solve this standard maximization problem using the simplex method. Start by introducing slack variables $s_1$ and $s_2$.

The first simplex matrix for the dual matrix will be:

\[
\begin{bmatrix}
  x_1 & x_2 & s_1 & s_2 & z \\
  2 & 1 & 1 & 0 & 0 & 3 \\
  3 & 4 & 0 & 1 & 0 & 2 \\
  -60 & -40 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

Make sure that you have negative values in your objective function.

Now pivot this matrix and write down the final simplex matrix. You will use only the original (4.2) rules to pivot, i.e. choose the column having the largest negative value in the bottom row first, then divide the RHS by each of the entries in the chosen column. Choose the lowest non-negative value to pick the pivot row. Pivot on the 3 in row 2, column 1.

Final matrix:

\[
\begin{bmatrix}
  x_1 & x_2 & s_1 & s_2 & z \\
  0 & -5/3 & 1 & -2/3 & 0 & 5/3 \\
  1 & 4/3 & 0 & 1/3 & 0 & 2/3 \\
  0 & 40 & 0 & 20 & 1 & 40
\end{bmatrix}
\]

This solution is optimal. The minimum value of $w$ is 40 and occurs when $y_1 = 0$, $y_2 = 20$ (you find these values in the last row under $s_1$ and $s_2$). The minimum value of $w$, 40, is the same as the maximum value of $z$.

**Further Uses of the Dual (page 178)**

**Shadow Values and Shadow Costs**

**Shadow Values**

When using duals with standard maximization problems, shadow values can be found by reading the values of $y_1$, $y_2$, $y_3$, etc. in the final simplex matrix. These shadow values indicate how much the maximized value would change for each unit increase or decrease of the indicated constraint in the original problem. Hopefully an example will help explain.

**Example:**

A baker has 150 units of flour, 90 units of sugar, and 150 units of raisins. A loaf of raisin bread requires 1 unit of flour, 1 unit of sugar, and 2 units of raisins, while a raisin cake needs 5, 2, and 1 units, respectively. If raisin bread sells for $1.75 per loaf and raisin cake for $4.00 each, how many of each should be baked so that the gross income is maximized? What is the maximum gross income?
$x_1 =$ number of loaves of raisin bread $x_2 =$ number of raisin cakes

Constraints:

1. $x_1 + 5x_2 \leq 150$ $s_1 =$ number of unused units of flour
2. $x_1 + 2x_2 \leq 90$ $s_2 =$ number of unused units of sugar
3. $2x_1 + 1x_2 \leq 150$ $s_3 =$ number of unused units of raisins
4. $x_1 \geq 0; x_2 \geq 0$ Objective function: $Z = 1.75x_1 + 4x_2$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$Z$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>150</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>90</td>
</tr>
<tr>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>150</td>
</tr>
<tr>
<td>-1.75</td>
<td>-4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Pivots

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.3</td>
<td>-0.3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-0.6</td>
<td>1.6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-3</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Reading the solutions off the final simplex matrix:

$x_1 = 50$, so the baker should make 50 loaves of raisin bread
$x_2 = 20$, so the baker should make 20 raisin cakes
$s_1 = 0$, so there are no unused units of flour
$s_2 = 0$, so there are no unused units of sugar
$s_3 = 30$, so there are 30 unused units of raisins

$y_1 = 0.16 = \frac{1}{6}$, so the shadow value of flour is $\frac{1}{6}$ per unit of flour

$y_2 = 1.583 = \frac{19}{12}$, so the shadow value of sugar is $\frac{19}{12}$ per unit of sugar

$y_3 = 0$, so the shadow value of raisins is 0

The shadow values are used to quickly determine how the maximum income is affected if the number of units of flour, sugar, or raisins available is changed without having to redo the matrices. This saves much more time than re-running the matrices if there are a large number of variables. Shadow values only give the exact answer for a limited range. You will not be required to determine this range.

To find the maximum income if the number of units of flour available is changed to 173, do the following:

Maximum income + (shadow value of flour)(change in availability of flour) = Adjusted maximum income

$167.5 + \left(\frac{1}{6}\right)(173 - 150) = 171.\overline{6} \approx 171.33$

To find the maximum income if the number of units of sugar available is changed to 80, do the following:

$167.5 + \left(\frac{19}{12}\right)(80 - 90) = 151.\overline{6} \approx 151.67$
To find the maximum income if the number of units of flour available is changed to 140, and the number of units of sugar available is changed to 110, and the number of units of raisins available is changed to 140, do the following:

\[
\$167.5 + \frac{1}{6} (140 - 150) + \frac{19}{12} (110 - 90) + 0(140 - 150) = \$197.50
\]

**Shadow Costs**

When using duals with standard minimization problems, shadow costs can be found by reading the values for \(x_1, x_2, x_3,\) etc. in the final simplex matrix. These shadow costs indicate how much the minimized value would change for each unit increase or decrease of the indicated constraint in the original problem. Hopefully an example will help explain.

**Example:**

Mr. Jones needs to supplement his diet with at least 50 mg of calcium and 8 mg of iron daily. The minerals are available in two types of vitamin pills, P and Q. Pill P contains 5 mg calcium and 2 mg iron, while Pill Q contains 10 mg calcium and 1 mg iron. If each P pill costs 3 cents and each Q pill costs 4 cents, how could Mr. Jones minimize the cost of adding the minerals to his diet? What would the daily minimum cost be?

\[
y_1 = \text{number of Pill P} \quad y_2 = \text{number of Pill Q}
\]

**Constraints**

\[
egin{align*}
5y_1 + 10y_2 & \geq 50 \\
2y_1 + y_2 & \geq 8 \\
y_1 & \geq 0, y_2 & \geq 0
\end{align*}
\]

**1st matrix**

<table>
<thead>
<tr>
<th></th>
<th>5</th>
<th>10</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>calcium</td>
<td>2</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>iron</td>
<td>.03</td>
<td>.04</td>
<td>0</td>
</tr>
</tbody>
</table>

**Transposed matrix**

\[
\begin{bmatrix}
5 & 2 & .03 \\
10 & 1 & .04 \\
50 & 8 & 0
\end{bmatrix}
\]

**Inequalities of the dual**

\[
5x_1 + 2x_2 \leq .03 \\
10x_1 + 1x_2 \leq .04 \\
w = 50x_1 + 8x_2
\]

Since the .03 and .04 represent dollars in the dual, in order for the units to be the same on both sides of the inequality, \(x_1\) and \(x_2\) must represent the cost in dollars of one mg of calcium and one mg of iron respectively.

To continue solving, set up the dual matrix and solve according to the simplex method:

\[
\begin{bmatrix}
5 & 2 & 1 & 0 & 0 & .03 \\
10 & 1 & 0 & 1 & 0 & .04 \\
-50 & -8 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

**Pivots**

<table>
<thead>
<tr>
<th></th>
<th>Pivots</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>x_1</td>
<td>x_2</td>
<td>s_1</td>
<td>s_2</td>
<td>Z</td>
<td>x_1</td>
<td>x_2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2/3</td>
<td>-1/3</td>
<td>0</td>
<td>1/150</td>
<td></td>
</tr>
<tr>
<td>R2 C1</td>
<td>0</td>
<td>1</td>
<td>-1/15</td>
<td>2/15</td>
<td>0</td>
<td>1/300</td>
</tr>
<tr>
<td>R1 C2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>11/50</td>
</tr>
</tbody>
</table>
Reading the solutions off the final simplex matrix:

\[ y_1 = 2, \text{ so Mr. Jones needs to take two Pill P's} \]
\[ y_2 = 4, \text{ so Mr. Jones needs to take four Pill Q's} \]

\[ Z = w = \frac{11}{50} = .22, \text{ so the minimum cost is } .22 \]

\[ x_2 = \frac{1}{150} = .006, \text{ so the shadow cost of iron is } .0067 \text{ per mg} \]
\[ x_1 = \frac{1}{300} = .003, \text{ so the shadow cost of calcium is } .0033 \text{ per mg} \]

The shadow costs are used to quickly determine how the minimum cost is affected if the number of mg of calcium or iron required is changed without having to redo the matrices. This is a much greater timesaver than re-running the matrices if there are a large number of variables. Shadow costs only give the exact answer for a limited range. You will not be required to determine this range.

To find the minimum cost if the number of mg of calcium required is changed to 53, do the following:

\[ \text{Minimum cost} + (\text{shadow cost of calcium})(\text{change in requirement}) = \text{Adjusted minimum cost} \]
\[ .22 + (.0033)(53 - 50) = .22 + .0099 = .2299 \approx .23 \]

To find the minimum cost if the number of mg of calcium required is changed to 48, do the following:
\[ .22 + (.0033)(48 - 50) = .22 - .0069 = .2134 \approx .21 \]

To find the minimum cost if the number of mg of calcium required is changed to 48 and the number of mg of iron required is changed to 9, do the following:
\[ .22 + (.0033)(48 - 50) + (.0067)(9 - 8) = .22 + .0069 + .0067 = .2336 \approx .23 \]