

What does ‘less than or equal’ really mean?

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Abstract

The Cantor-Bernstein theorem is often stated as ‘ $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{b} \leq \mathfrak{a}$ imply $\mathfrak{a} = \mathfrak{b}$ ’ for cardinalities. This suggestive form of the theorem may lead to a trap, into which many early twentieth century mathematicians fell, unless we are very careful in interpreting \leq . The key is the subtle interplay between $<$ and \leq . Originally, following Cantor, $<$ was considered the primary relation, and \leq was defined as the disjunction of $<$ and $=$. However, the above suggestive form of the Cantor-Bernstein theorem requires the modern definition of \leq . The uncertainty, sometimes confusion, and evolution due to these subtleties can fascinate and motivate both us and our students today.

Today we compare cardinalities by simple containment. For sets A and B , we define $\mathfrak{a} \leq \mathfrak{b}$ to hold between their respective cardinalities if and only if A is equivalent¹ to a subset of B . We imagine that this natural notion was always just so, but it was not, and therein hangs a tale of uncertainty, sometimes confusion, and evolution over several decades of foundational subtleties in mathematics.

Consider what Felix Hausdorff wrote in 1927:

The Equivalence Theorem can then be written in the suggestive form: If $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{a} \geq \mathfrak{b}$, then $\mathfrak{a} = \mathfrak{b}$. [18, p. 29] [19, p. 32]

This sentence seems at first sight eminently reasonable and clear, given Hausdorff’s statement of the Equivalence Theorem (today called the Cantor-Bernstein Theorem or Schröder-Bernstein Theorem²):

If each of two given sets is equivalent to a subset of the other, then the two given sets are themselves equivalent. [18, p. 27] [19, p. 30]

Surprisingly, though, this simple interpretation is nixed by the discovery that what Hausdorff meant by \leq is not what we mean today! Rather, he had defined it in part of an earlier sentence:

If A is equivalent to a subset of B (case (1) or (2)), then $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} < \mathfrak{b}$, which we write as the single expression $\mathfrak{a} \leq \mathfrak{b}$. [18, p. 29] [19, p. 32]

¹Recall that two sets X and Y are equivalent provided they can be put in one-to-one correspondence.

²While Georg Cantor proved the theorem in special situations, Richard Dedekind actually had the first general proof in 1887, which he described more than a decade later in their 1899 correspondence [15, p. 449], even though he never published it. This nontrivial theorem received many different proofs. For a detailed history see [20, 25, 26].

Thus for Hausdorff $\mathfrak{a} < \mathfrak{b}$ was more fundamental than $\mathfrak{a} \leq \mathfrak{b}$, used in a disjunction to define \leq , quite unlike our approach today. So to understand what \leq meant to him, we must first learn his meaning for $<$, contained in the following excerpt (where \sim stands for equivalence).

If A and B denote sets, and A_1 and B_1 any subsets of A and B respectively, then there are four [mutually exclusive] possibilities:

- (1) There is an $A_1 \sim B$, and a $B_1 \sim A$.
- (2) There is no $A_1 \sim B$, but there is a $B_1 \sim A$.
- (3) There is an $A_1 \sim B$, but no $B_1 \sim A$.
- (4) There is no $A_1 \sim B$, and no $B_1 \sim A$.

In case (1) we have, by the Equivalence Theorem, $\mathfrak{a} = \mathfrak{b}$; in case (2) it is natural to define $\mathfrak{a} < \mathfrak{b}$, and in case (3) to define $\mathfrak{a} > \mathfrak{b}$. In case (4), ... [we ...] have a fourth relation, which we write as $\mathfrak{a} \parallel \mathfrak{b}$, and call incomparability between \mathfrak{a} and \mathfrak{b} , while in the first three cases \mathfrak{a} and \mathfrak{b} will be called comparable. [18, p. 28] [19, p. 31f]

This settles Hausdorff's definition of $<$. Now, finally, we can interpret what he says about \leq . He defines $\mathfrak{a} \leq \mathfrak{b}$ to mean " $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} < \mathfrak{b}$ ". But then from $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{a} \geq \mathfrak{b}$ it trivially follows that $\mathfrak{a} = \mathfrak{b}$, from the mutual exclusivity of $\mathfrak{a} < \mathfrak{b}$ and $\mathfrak{a} > \mathfrak{b}$. This seems absurd, Hausdorff claiming the Equivalence Theorem is a trivial consequence of a natural definition. Something is wrong here.

If we want to write the Equivalence Theorem in Hausdorff's suggestive form, then we have to define $\mathfrak{a} \leq \mathfrak{b}$ differently, as " A is equivalent to a subset of B ", as is done today. The snag here is that the logical equivalence of the two statements " A is equivalent to a subset of B " and " $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} < \mathfrak{b}$ " relies on the Equivalence Theorem itself, as Hausdorff even said above when addressing case (1). So to use this logical equivalence to create his suggestive form for the Equivalence Theorem is deceptive at best, circular at worst.

It turns out Hausdorff was far from the first to fall into this trap. Philip Jourdain in 1907 [22, pp. 355–356] had already explained what is really going on here. He named who had already stumbled, and he suggested a way of avoiding confusion about the meaning of \leq and the role of the Equivalence Theorem.

It is important to be quite clear as to which is proved in this theorem. Bernstein, Whitehead, König, and Peano, owing to what is, I think, a misleading use of the symbol (\leq), have given rise to much confusion. These writers, namely, state the theorem in question as follows:

If $m \leq n$ and $m \geq n$, then $m = n$.

If $m \geq n$ means, as it is natural to suppose it does, ($m > n$ or $m = n$), this statement is a simple logical conclusion from the fact that each of the relations $>$, $<$, $=$ excludes the other two, and hardly needs a special 'theorem' in mathematics. But Bernstein, for example, seems to mean by $m \leq n$ the statement 'there is a part of N which is similar to M ,' and thus either case (1) or (3) occurs.³ By Cantor's definitions⁴ of $<$

³Jourdain's case (3) is case (2) in Hausdorff's listing, with $M = A$, $N = B$, $m = \mathfrak{a}$, and $n = \mathfrak{b}$.

⁴Hausdorff's definition of $<$ is adopted from Cantor.

and $=$, we can state from this that ‘ $m < n$ or $m = n$ ’ when, and when only, we have proved the Schröder-Bernstein theorem.

It seems less misleading to give (\leq) the natural signification and to state the Schröder-Bernstein theorem (in the form it takes when M and N have cardinal numbers) as follows:

‘If a part of N is similar to M , then

$$m \leq n.’$$

Even earlier, in 1898, when Émile Borel published Felix Bernstein’s beautiful proof of the Equivalence Theorem in his book [9, pp. 106–107], he warned about this trap and how it is a dangerous artifact of our use of language:

We will especially use this [theorem] in the following form: When an aliquot part of B has the same power as A , we say that *the power of A is less or equal to that of B and that the power of B is greater or equal to that of A* .⁵ From then on, when we have shown that the power of a set A is both *greater or equal* and *less or equal* to that of a set B , we can affirm that *it is equal to it*. In this form, the theorem appears obvious; but it is important to notice that if it had not been demonstrated beforehand it would not be legitimate to introduce this language.

To summarize, there are two possible definitions of \leq . One is based on Hausdorff’s definition of $<$ and is given by

$$a \leq b \text{ if and only if } a < b \text{ or } a = b.$$

The other definition is the modern one, and is given by

$$a \leq b \text{ if and only if there is a subset of } B \text{ that is equivalent to } A.$$

Clearly the disjunctive definition of \leq implies the modern one. The converse requires the Equivalence Theorem. Notice, in fact, that Jourdain ends by correctly phrasing the Equivalence Theorem as saying that the modern meaning of \leq implies the disjunctive meaning.⁶ But Jourdain does not recommend adopting our modern definition of \leq . This leaves us wondering who first committed to the modern approach of introducing \leq as the primary relation, without first introducing $<$, and why. The earliest reference we have found that gives the modern definition of \leq is Giuseppe Peano’s 1906 reformulation of the Equivalence Theorem [27]. The earliest textbooks we have found taking the modern approach are the 1939 summary *Théorie des ensembles (Fascicule de résultats)* by Bourbaki [10, p. 38], precursor to its appearance in the 1956 Bourbaki textbook *Théorie des ensembles* [11, p. 56]

⁵There appears to have been a typographical error here: A and B were switched in the first clause, which we have reversed above.

⁶Waclaw Sierpinski discusses this extensively in his book [30, p. 80f] on cardinal and ordinal numbers. In fact in the many editions of his books on set theory and cardinals, beginning with [29, p. 72f, 81f] in 1912, Sierpinski defines $<$ first, then correctly deals with the tricky issue in defining and using \leq (his 1910 handwritten set theory notes [28] do not yet discuss \leq).

[12, p. 158f], and the 1941 *A survey of modern algebra* by Birkhoff and MacLane [8, p. 339]. Since then it has become standard to define \leq first.

But if we today define \leq first, this leaves open the question of what $<$ should mean. Let us look at how some modern textbooks deal with this issue, having defined \leq as the primary relation. There are two alternatives here too.

In Jech [21, p. 23] and Kuratowski-Mostowski [24, p. 185] we see $<$ defined, perfectly naturally, as

$$\mathfrak{a} < \mathfrak{b} \text{ if and only if } \mathfrak{a} \leq \mathfrak{b} \text{ and } \mathfrak{a} \neq \mathfrak{b}.$$

However, notice that this natural definition is *not* Hausdorff's definition (his case (2)), which is unsettling.

On the other hand, in Kunen [23, p. 27] we find $<$ defined as

$$\mathfrak{a} < \mathfrak{b} \text{ if and only if } \mathfrak{a} \leq \mathfrak{b} \text{ and } \mathfrak{b} \not\leq \mathfrak{a},$$

which *is* Hausdorff's definition.

In fact, Hausdorff's definition of $<$ was taken from Cantor's famous 1895 paper [13] [14, p. 89f]. In contrast, though, the natural definition above of $<$ was actually Cantor's first attempt, in 1878 [15, p. 119]:

If the two sets M and N are not of the same power, then either M has equal power with a component part of N , or N has equal power with a component part of M ;⁷ in the first case we call the power of M *smaller*, in the second case we call it *larger*, than the power of N .⁸

If we define $<$ using this natural definition, we find that its antisymmetry (that $\mathfrak{a} < \mathfrak{b}$ and $\mathfrak{b} < \mathfrak{a}$ cannot both occur) and transitivity ($\mathfrak{a} < \mathfrak{b}$ and $\mathfrak{b} < \mathfrak{c}$ imply $\mathfrak{a} < \mathfrak{c}$) are not obvious without the use of the Equivalence Theorem. On the other hand, if $<$ is defined as Cantor did in 1895, then antisymmetry and transitivity of $<$ are straightforward.

In 1887 Cantor wrote [15, p. 413].

If by some means it is determined, that two given sets M and N are not equivalent, then one of the following two cases occurs: either a component part N' can be separated from N , so that $M \sim N'$, or a component part M' can be separated from M , so that $M' \sim N$. In the first case $\overline{\overline{M}}$ ⁹ is called smaller than $\overline{\overline{N}}$, in the second we call $\overline{\overline{M}}$ larger than $\overline{\overline{N}}$.

⁷Surprisingly, by "part" Cantor means proper subset, something we did not initially realize, which led to much confusion on our part. On the other hand, Cantor's contemporary Dedekind made a clear distinction between subset and proper subset by calling them "part" and "proper part." In their correspondence, they needed to acknowledge their different meanings for "part" since in some situations the distinction is highly relevant. For example, in a 1899 letter to Dedekind, Cantor writes "When I speak of 'part', I always mean 'proper part'." [15, p. 450] For this particular definition, however, the distinction does not influence the meaning.

⁸Notice here Cantor's implicit belief that the powers (cardinal numbers) of any two sets are inherently comparable. This is not necessary for the definition being made.

⁹The notation $\overline{\overline{M}}$ is Cantor's terminology for the cardinal number of the set M .

Here it cannot be sufficiently emphasized, that the exclusive behavior of the two cases, which underlies the definition of greater and lesser for cardinal numbers, depends vitally on the assumption made, that M and N do not have the same power. If the two sets were equivalent, it could certainly happen, that component parts M' and N' exist, for which both $\overline{\overline{M}} = \overline{\overline{N'}}$ and $\overline{\overline{M'}} = \overline{\overline{N}}$. One has the theorem: if M and N are two such sets, from which component parts M' and N' can be separated, for which one can show that $\overline{\overline{M}} = \overline{\overline{N'}}$ and $\overline{\overline{M'}} = \overline{\overline{N}}$, then M and N are equivalent sets.

This represents transitional thinking between what Cantor wrote in 1878 and 1895, since he recognizes here that antisymmetry of the natural definition of $<$ depends on the Equivalence Theorem. It is interesting to note how in Cantor's writings the Equivalence Theorem took on increasing significance with time [15, pp. 201, 257].

In 1895 Cantor changed his definition of $<$ [14, p. 89f].

If for two sets¹⁰ M and N with the cardinal numbers $\alpha = \overline{\overline{M}}$ and $\mathfrak{b} = \overline{\overline{N}}$, both the conditions:

- (a) There is no part of M which is equivalent to N ,
- (b) There is a part N_1 of N , such that $N_1 \sim M$,

are fulfilled, it is obvious that these conditions still hold if in them M and N are replaced by two equivalent sets M' and N' . Thus they express a definite relation of the cardinal numbers α and \mathfrak{b} to one another.

...

We express the relation of α to \mathfrak{b} characterized by (a) and (b) by saying: α is "less" than \mathfrak{b} or \mathfrak{b} is "greater" than α ; in signs

$$\alpha < \mathfrak{b} \text{ or } \mathfrak{b} > \alpha.$$

According to Hinkis [20, §5.1], Cantor's change in his definition of $<$ was precisely because he had difficulties giving an elementary proof that his 1878 definition is transitive. Another reason could be that the 1895 definition fits perfectly as one of the four mutually exclusive cases for comparing sets (Hausdorff's case (2)). These four cases were discussed by Cantor in a 1899 letter to Dedekind [15, p. 450] (see also [9, p. 102ff]).

It is clear that Cantor's 1895 definition implies the 1878 definition, while the converse needs the Equivalence Theorem. Indeed in 1942 Paul Bernays [6] uses Cantor's more natural 1878 definition of $<$, rather than the 1895 definition, but only after establishing the Equivalence Theorem first.

¹⁰This translation is by Jourdain. He calls sets 'aggregates,' but we choose to use 'set' instead since it is by now an established word.

Confusion, resolution, and pedagogy

The seemingly simple question of how to define and work with inequality ($<$) and less than or equal (\leq) for cardinalities led to over half a century of uncertainty. Moreover, it sometimes led to confusion, and evolution in research and textbooks, by some of the best mathematicians of the era, due to several wonderful foundational issues that are eminently accessible to undergraduate students, and that can fascinate us today. The definition of $<$ evolved even in Cantor's own writings over a 17 year period as he grappled with subtleties of the situation.

Originally $<$ was considered as the primary relation, and \leq was defined as the disjunction of $<$ and $=$. Then the statement

$$\mathfrak{a} \leq \mathfrak{b} \text{ and } \mathfrak{a} \geq \mathfrak{b} \text{ imply } \mathfrak{a} = \mathfrak{b}$$

is trivially true. On the other hand, under the modern definition, the roles of the two relations are reversed. The relation \leq is primary, and $<$ is defined as the conjunction of \leq and either \neq or $\not\geq$ (which are equivalent definitions by the Equivalence Theorem). The implication above then becomes a restatement of the Equivalence Theorem. The original and modern definitions of \leq are equivalent by the Equivalence Theorem, but the modern approach has the advantage that many developments and proofs are streamlined.

Today's textbooks sweep these subtleties under the rug via the modern definition of \leq , and present a development that hides the foundational issues from both us and our students. But these very issues provide the richness and inherent interest that are at the core of what mathematics is, that we should want for our students, and that will help them think deeply about mathematics [3, 4].

In our own teaching, we have implemented a pedagogical philosophy of having students acquire fundamental mathematical ideas, knowledge, and practice by studying excerpts from primary historical sources [1, 2, 5, 7]. In teaching discrete mathematics, our intention was to teach infinity, cardinals, and \leq from primary sources. To our amazement, we discovered that the earlier sources did not even define \leq , but rather focused on $<$. Thus began our own journey that stimulated us and our students to think more deeply about matters that today's mathematicians imagine are straightforward, and to find hidden gems of new interest and insight. In fact, primary sources often provoke many natural and important questions about definitions. Through guided reading and study of carefully selected primary source excerpts, our students (and we) can benefit from confronting and resolving conceptual issues, precisely because the sources have not been retrospectively sanitized to remove the issues and questions that got 'cleared up' later, and then often lost.

ACKNOWLEDGEMENTS. We thank Patricia Baggett, Nick Bezhaniashvili, Andrzej Ehrenfeucht, Tiziana Giorgi, Tadeusz Litak, and Wacek Zawadowski for suggestions, assistance with translation and obtaining sources. We also thank interlibrary loan at New Mexico State University.

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