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The Pedagogy of Primary Historical Sources in Mathematics

Classroom Practice Meets Theoretical Frameworks

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Abstract We analyze our method of teaching with primary historical sources within the context of theoretical frameworks for the role of history in teaching mathematics developed by Barbin, Fried, Jahnke, Jankvist, and Kjeldsen and Blomhøj, and more generally from the perspective of Sfard's theory of learning as communication. We present case studies for two of our guided student modules that are built around sequences of primary sources and are intended for learning core curricular material, one on logical implication, the other on the concept of a group. Additionally, we propose some conclusions about the advantages and challenges of using primary sources in teaching mathematics.

Keywords mathematics, history, curricular modules, primary sources, guided reading, undergraduate

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1 Introduction

During twenty-five years of teaching with primary historical sources, we have undergone a largely subconscious coevolution between the details of *how* we teach with primary sources and *why* we teach with primary sources. This interplay is now being consciously stimulated by the influence and continuing development of various theoretical frameworks for the role of history in teaching mathematics. Our goal in this paper is to examine and compare the dynamic interaction between our practical hows and whys and various theoretical analyses, and to propose some conclusions about the advantages and challenges of teaching with primary sources by comparing the results of our practical teaching experiences with some of these theoretical frameworks.

We begin in the next section with the story of our own experiences with primary sources in the undergraduate mathematics classroom. This recounting of our journey highlights the ways in which we have employed primary sources to promote students' understanding of mathematics, while postponing discussion of the historical concerns and theoretical issues associated with those pedagogical practices. The subsequent section summarizes key features of several theoretical frameworks which are currently influencing the way in which we think about our classroom practice; namely, the application of Sfard's theory of thinking as discourse, the notions of metadiscursive rules and commognitive conflict as these have been developed by Jankvist, Kjeldsen, and Blomhøj, and the analyses of Barbin, Fried, and Jahnke on the advantages, opportunities, challenges, and imperatives in teaching modern mathematics and simultaneously authentically engaging primary sources on their own terms. Using evidence from student work, student comments, and the nature and design of our materials for students, we next offer two case studies, one on logical implication and one on the group concept, to illustrate the connection between our practical experience and these theoretical frameworks. We close the paper by discussing the interplay between our pedagogical approach and the theoretical frameworks.

2 A quarter century of interplay between our hows and whys teaching with primary sources

When some of us began teaching with primary sources, we at first simply provided our students with raw primary texts (in English translation if possible), supplemented with some lecture on historical context and biography, some mathematical pointers regarding the texts, then guided class discussion on the mathematics of the texts, followed by mathematical homework assignments to students.

As this commenced in the late 1980s, initially in honors courses created for this purpose with mathematical content entirely chosen by us via the texts selected, we were quite unaware of any theoretical frameworks for what we were doing. We were inspired by a paper of William Dunham (1986) on a historical great theorems course he conducted, in which he wrote and provided secondary materials, not primary sources, to his participants. We decided to create courses to provide a similar enriching experience learning mathematics from its history for university honors students, with the main difference being that we gave our students actual primary sources, rather than writing secondary materials replacing these sources. While we had the definite belief that it would be good for our students to grapple with primary sources, part of our motivation for giving primary sources directly to our students was that they were already written and available, whereas for us to write up secondary material based upon them would be much more work.

These first few years of classroom experiences spurred our pedagogical conceptions about why having students study primary sources directly is good for their learning. We elucidated these initial observations in articles such as Laubenbacher and Pengelley (1992), Laubenbacher and Pengelley (1996) and Laubenbacher et al (1994). One finds in Laubenbacher et al (1994), for example, the following perceived benefits for students of studying primary sources:

- Motivate abstract concepts.
- See the creative, artistic aspect, and intellectual fascination of mathematics.
- Witness mathematicians struggle, see the nature of mathematical practice and tradition, i.e., research, publication, discussion.
- Sequence of sources showing a chain of attempts to solve a problem, seeing the obstacles that need clearing.
- Bring students close to the experience of mathematical creation, false starts, triumphs.
- See the roots of modern problems, ideas, concepts.
- See the direction of mathematical development, flow, failures and successes.

Over a number of years we extended the materials for these special courses to annotated guided sequences of sources for nine mathematical themes, each published as a chapter in one of two books, *Mathematical Expeditions: Chronicles by the Explorers* (Laubenbacher and Pengelley, 1998) and *Mathematical Masterpieces: Further Chronicles by the Explorers* (Knoebel et al, 2007). Each chapter's theme was laid out as a grand story leading from past to present, with each section of each theme built around an annotated source, and then mathematical exercises for students. The goal of every book chapter was to learn and do specific mathematics while experiencing the evolution of the solution of one particular problem or challenge over centuries or millennia through a sequence of sources, each source displaying a contribution to the on-going effort to resolve the problem, leading to an eventual sense of satisfaction at the modern state of affairs.

Beginning around 2003 a still expanding team of faculty began a quite different program, to incorporate primary sources into the syllabi of regular courses in the curriculum, initially in discrete mathematics and computer science. A collaboration of first five and then seven mathematics and computer science faculty, supported by the US National Science Foundation, developed materials based on primary sources for core material in courses on discrete mathematics, abstract algebra, algorithms, automata and formal languages, calculus, combinatorics, data structures, graph theory, mathematical logic, programming languages, and theory of computation[1] (Barnett

[1] Seventeen of our most recent projects for students will be forthcoming as articles in the online Convergence: Loci journal of the Mathematical Association of America.

et al, 2004–, 2008–, 2009). The testing and dissemination of these materials involved numerous other faculty as teachers and new authors, and currently an evolving team with both old and new team members is preparing to develop such materials for several other courses for mathematics and mathematics education majors.

As we carried this out, we embedded primary sources into a "guided reading module" or "student project" format, for flexible insertion by an instructor into a course, sometimes intended to replace regular textbook material. The individual modules are limited in scope for flexible classroom use, usually utilizing only one or a very few primary source excerpts. Nonetheless, once enough of these modules were created, we have been able to teach entire courses (Discrete Mathematics, Combinatorics, Mathematical Logic) from them, dispensing with standard textbooks.

The core of a guided reading module is a selection of excerpts from original sources related to the evolution of the mathematical concepts in question, together with a series of tasks designed to illuminate the source material and prompt students to develop their own understanding of the underlying concepts and theory. Each module also includes a discussion of the historical context and mathematical significance of each primary source selection, and brief biographical information on the authors of those texts. In contrast to the standard practice of placing exercises only at the end of each section or chapter, these tasks require students to actively engage with the mathematics as they read and work through each excerpt. Some of these tasks extend the original source excerpts by prompting students to fill in missing proof details or to reflect on the standards of rigor and style of proof illustrated by the excerpt. Through these, students' ability to construct proofs in keeping with present-day standards can be progressively extended. Students are also introduced to present-day notation, definitions, and terminology at appropriate junctures in their reading of the original sources, either through tasks or in the secondary commentary.

Note that the pedagogy that evolved for these guided reading modules contains some very substantial shifts from the approach of our first years teaching with primary sources in honors courses. In a guided reading module students are led via frequent tasks tied to explicitly selected spots in the source texts to develop their own understanding of concepts and theory, and to reflect from the primary source on the evolution of standards of rigor, and of the nature of proof, definitions, notation, terminology, etc. Thus, in addition to the general benefits of primary sources identified much earlier, which fall more within the (passive) "motivate, see, witness" categories, the new benefits now explicitly incorporated in student tasks in our guided reading modules are of another nature. In our recent paper Barnett et al (2011), we identified, among others, the following additional design goals for our guided reading modules, going beyond the earlier "motivate, see, witness" category. Although we presented these as design goals, they really display additional "whys" for teaching with primary sources, whys of which we had slowly become aware as they emerged from student responses to our materials and which, in turn, we had begun to build more explicitly into our own written materials for student learning:

- Hone students' verbal and deductive skills through reading.
- Provide practice moving from verbal descriptions of problems to precise mathematical formulations.

- Promote recognition of the organizing concept behind a procedure.
- Promote understanding of the present-day paradigm of the subject through the reading of an historical source which requires no knowledge of that paradigm.
- Promote reflection on present-day standards and paradigm of subject.
- Draw attention to subtleties, which modern texts may take for granted, through the reading of an historical source.
- Encourage more authentic (versus routine) student proof efforts through exposure to original problems in which the concepts arose.
- Engender cognitive dissonance (*dépaysement*) when comparing a historical source with a modern textbook approach, which to resolve requires an understanding of both the underlying concepts and use of present-day notation.

In short, the primary source is now being used not just to introduce the mathematics in an authentically motivated context, but also as a text which the student is explicitly challenged to actively "interpret" as part of their personal process of making modern mathematics their own. In alignment with this shift, the tasks we now write for students increasingly adopt a more active "read, reflect, respond" approach to these sources. In recent years, our own understanding of the pedagogical value of such tasks, and of the many issues, challenges, and pitfalls in teaching with primary sources, has been influenced by the theoretical analyses of others in ways that we will discuss and reflect upon in the rest of the paper.

3 Theoretical connections

The body of literature on the use of history of mathematics in mathematics education has expanded significantly over the past quarter century. In this section, we briefly describe a selection of theoretical and research works from the literature which have come to influence our thinking about both why and how we teach with primary sources in the ways we do.

We begin with Jankvist (2009), a work which underscores the value — both for research and for curriculum development — of clearly distinguishing between, and also explicitly stating, the "hows" and "whys" which frame any particular pedagogical use of history. Of the three approaches described by Jankvist concerning *how* one might bring an historical point of view to the classroom — "illuminations," "modules" and "historical" — our own approach clearly falls into the second category. Jankvist further proposes a two-category system for classifying the various reasons *why* such a historical point of view might be adopted in mathematics education. Within the "history-as-a-goal" category are those arguments in favor of using history which concern the teaching of metaperspective issues (or metaissues) of mathematics; for example, uses of history aimed at raising students' awareness of the human and cultural aspects of mathematics and how these have shaped the development of the discipline. In contrast, "history-as-a-tool" arguments concern the teaching and learning of the inner issues (e.g., concepts, theories, techniques, algorithms) of mathematics.

Our own current efforts to use history as a cognitive support for learning mathematical concepts, theories and methods in an active "read, reflect, respond" fashion are thus driven by goals which fall into the history-as-a-tool category. Despite the change in *how* we now seek to achieve the educational goals behind our earlier "motivate, see, witness" approach to using original sources, the intention of supporting student learning of inner issues by using history to provide context, motivation and direction to their mathematical studies remains with us, thereby constituting an additional set of goals which fall primarily in the history-as-a-tool category. Jankvist's categorization scheme has made us aware, however, that certain history-as-a-goal objectives have become an important part of our work (albeit to differing degrees for those of us involved in this work). For example, in addition to the history-as-a-tool objectives already listed in Section 2, the following history-as-a-goal objectives appeared in the list of our pedagogical design goals in Barnett et al (2011).

- Promote a human vision of science and of mathematics.
- Promote a dynamical vision of the evolution of mathematics.
- Promote enriched understanding of subject through greater understanding of its roots, for students and instructors.

As Jankvist observes, history may only be an indispensable component of mathematics teaching with respect to history-as-a-goal objectives such as these, with other pedagogical approaches (e.g., applications of mathematics) serving at least as well (Jankvist, 2009, p. 245) for history-as-a-tool objectives. One intention of his proposed categorization system is to encourage research into such questions by providing a means for their proper framing. For practitioners such as ourselves, his proposed categorization also highlights the challenges of evaluating our student projects relative to our goals for students.

A different type of challenge concerning both our goals and *how* we approach them is posed by Fried's suggestion that only those historical approaches aimed at objectives which align with what he calls the "cultural theme," such as the three objectives just listed, allow mathematics educators to take history "seriously *as a form of knowledge*" (Fried, in press, p. 32). In this and earlier works (e.g. Fried, 2001, 2007), Fried has written passionately about the inherent tension which exists between the goals of mathematics educators who seek to use history as a tool for developing mastery of the techniques and applications of modern mathematics, and the historian's commitment to "shun anachronism, to see mathematics as something in flux, and, therefore, to understand past mathematics as something more than an older version of present mathematics" (Fried, 2001, p. 405). Although we embrace the commitment to humanizing mathematics which Fried contends requires one to take history seriously by "look[ing] at [mathematics] through the eyes and works of its practitioners, with all their idiosyncrasies" and "as far as possible, [to] read *their* texts as *they* wrote them" (Fried, 2001, p. 401), our own professional obligations as instructors of mathematics commit us to teaching within the framework of a modern undergraduate mathematics curriculum. Our intentions to use history as a cognitive tool in support of the objectives of that curriculum thus places our work directly at the center of this tension, as displayed in Figure 5 of Fried (2007, p. 218) where a "mathematics educator" is a bridge between a "historical way of knowing mathematics" and a "working mathematician's way of knowing mathematics."

By guiding students through the reading of original source material, our classroom projects also fall short of Fried's call to adopt a "radical accommodation" strategy of (directly) studying (original) mathematical texts as a means of avoiding the danger of trivializing history by adopting a "Whig" approach when using it as a teaching tool (Fried, 2001).[2] His eloquent discussion of the differences between a "text" and a "textbook" nevertheless encapsulates one of the essential qualities which we feel original source readings have to offer as a cognitive support to students:

> [...] textbooks and texts are different precisely because the latter *are* original, where being original has not to do with being old but with the immediacy of the author's engagement with his subject. Because of this difference, readers of textbooks and texts will be led in different directions. For when a book merely sets out accepted knowledge, as a textbook must do almost by definition, it must essentially be closed to inquiry. This is so even when such a book asks questions about the material it contains, for the intention of such questions is only to clarify what is finally to be accepted. For inquiry to arise, what one faces must be, in a sense, conditional, open, uncertain; one must be in the position of asking not so much whether what one is reading is clear and understood (as important as that may be), but whether it is true. This is the position of a thinker in an original encounter with his subject and, to the extent that the author is a great one, that is, is wholly engaged in the kind of inquiry such an original encounter requires, the reader is drawn into the inquiry and invited to search for the "unsedimented" meaning of every word. (Fried, 2001, pp. 402-403).

Fostering the type of *radical engagement* with the historical author's original inquiry suggested here is precisely the type of experience which our guided reading modules seek to offer students.

Put somewhat differently, and borrowing a phrase from Jahnke et al (2002), much of the educational value of original source readings in any discipline lies in their ability to promote "thinking into other persons and into a different world" (Jahnke et al, 2002, p. 299). For mathematics education in particular, radical engagement with original sources can provide cognitive support to students in part because "she who thinks herself into a scientist doing mathematics at a different time has herself to do mathematics" (Jahnke et al, 2002, p. 299). Given that what it means "to do mathematics" depends in part on one's historical context, Jahnke et al (2002) rightfully stress the need for texts to be located in the context of their time. They further repeat an earlier warning against the danger of a teleological reading sounded by Barbin, who observed in particular that the educational value of the *reorientation* (or *dépaysement*) which reading historical texts can effect in our view of mathematics by "making the familiar unfamiliar" requires a reading which seeks always to understand, rather than judge, the content of those texts (Barbin, 1997, p. 24). The role of *replacement*, an effect which arises when the "usual" is replaced by something "different" through the reading of original texts, in promoting a view of mathematics as an intellectual activity, rather than a mere body of knowledge or a set of techniques, is also discussed in both Barbin (1997) and Jahnke et al (2002). Related to this is the observation that

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[2] Radical accommodation is one of two alternatives that Fried identifies as a means to avoid this danger; the second alternative is that of "radical separation" in which the study of the history of mathematics is placed on an entirely different track from the regular course of study.

"thinking themselves into other persons motivates students to reflect about their own views of the subject matter" (Jahnke et al, 2002, p. 299).

In summary, Fried, Jahnke, and Barbin, among others, have articulated theoretical perspectives which suggest that the proper use of original sources may achieve both history-as-a-goal and history-as-a-tool objectives. These particular perspectives also align well with our own broad view of what it means to be "mathematically educated" as reaching beyond mere technical ability to include an understanding of what mathematics itself is, both in its traditions and in the general underlying methods which govern the way in which the discipline is (or can be) practiced. Many of our student projects, for example, seek to use original sources as a means to promote reflection on present-day standards of rigor, communication and proof by explicitly drawing students' attention to subtleties which modern textbooks may take for granted. At the same time, the readings themselves provide students with a basis for their reflections, thereby helping them to see how to develop ideas and reason with them on their own.

Additional arguments concerning the ways in which original source readings can help students develop mathematical competencies such as the ability to develop and reason with ideas on their own are found in research conducted by Jankvist, Kjeldsen and Blomhøj (e.g. Jankvist, 2010; Jankvist and Kjeldsen, 2011; Kjeldsen and Blomhøj, 2011) which draws extensively on Sfard's theory of mathematical discourse. In the subsequent two sections of this paper, we provide case studies which examine how Sfard's theoretical framework also connects with our classroom practice. To set the stage for this discussion, we close this section with a brief description of Sfard's theory, drawing primarily on Sfard (2008).

Adopting the view that learning entails becoming a participant in a certain discourse (versus a view of learning as the acquisition of knowledge), Sfard defines "thinking" as "an individualized version of (interpersonal) communication" (Sfard, 2008, p. 81). To stress the combination of "communication" and "cognition" inherent in this definition, she coins the term "commognition" as a contraction of these two terms. As in all human communication, any particular discourse is then an activity regulated by rules at two different levels, with object-level rules depicting regularities in the behavior of the discursive objects, while metadiscursive rules reflect the regular nature of the discourse itself. In mathematics, for example, it is the metadiscursive rules which govern mathematical practices such as determining what constitutes a proper definition, or deciding whether a solution is correct and complete.

While sharing those features common to all communities of discourse, any individual community is characterized by the distinctive discourse which it creates; in the case of a mathematical community, according to Sfard, the distinguishing characteristic of its discourse lies in mathematics' status as an "autopoietic system" which is self-generative in the sense that it produces its own conceptual referents; that is, "a system that contains the objects of talk along with the talk itself and that grows incessantly 'from inside' when new objects are added one after another" (Sfard, 2008, p. 129).

One consequence of the autopoietic nature of mathematical discourse is the relative nature of the distinction between object-level rules and metadiscursive rules, with metarules in one mathematical discourse giving rise to an object-level rule "as soon as the present metadiscourse turns into a full-fledged part of mathematics itself"

(Sfard, 2008, p. 202). As in other discourse domains, however, metadiscursive rules in mathematics are constraining (versus deterministic), contingent (versus necessary), and mainly tacit. As a consequence, Sfard notes, the "rules of language games can only be learned by actually playing the game with experienced players. Thus, it is naive to think that either mathematical discursive habits or the ability to speak a foreign language could be developed by [students] left to themselves" (Sfard, 2000, p. 185).

Sfard further proposes that "commognitive conflict" — that is, a conflict marked by a participant's encounter with two incommensurable discourses — is practically indispensable for learning at the metalevel. A commognitive conflict, for example, may arise when a student who is conversant in the discourse of school mathematics encounters the quite different metalevel rules which govern the discourse practice of a university classroom. The resolution of such a conflict is marked by one participant's acceptance of the discursive rules of the other; in this example, the student's acceptance of the discursive rules of the university professor who serves as the "expert interlocutor."

With respect to the use of history in mathematics education, Kjeldsen and Blomhøj have suggested that the reading and interpretation of original sources may be essential for raising student's awareness of the "meta-discursive rules" which govern (current) mathematical practices in order to engender the commognitive conflict that, in turn, would lead to metalevel learning (Kjeldsen and Blomhøj, 2011). The case studies we offer in the next two sections of this paper seek to illustrate how this particular suggestion and Sfard's theoretical framework more generally connect with our current approach to teaching with original sources.

4 A Curricular Module Case Study: *Deduction Through the Ages*

The historical modules we have developed are explicitly designed to explore the origins behind modern definitions, lemmata and algorithms, and offer context to these constructions well beyond a simple summary statement of the results, as often appears in modern textbooks. The module *Deduction Through the Ages* (Barnett et al, 2008–; Lodder, to appear) presents excerpts from several historical sources from antiquity to the twentieth century that offer insight into how modern mathematics arrived at the truth table of an implication (an "if-then" statement) in propositional logic. Many introductory textbooks in discrete mathematics contain a chapter on elementary propositional logic that begins with truth tables and the table for an implication $p \to q$ (p implies q). The truth values of the implication when the hypothesis, p, is false are counter-intuitive and most textbook authors offer some brief explanation of this, although these perfunctory remarks cannot do justice to over two millennia of philosophical thought on deductive reasoning, and students, when asked about the textbook's explanation, state only that it is confusing and perhaps the table is wrong. Nevertheless, students are willing to memorize the truth table of an implication, certainly if required for an examination.

The very fact that the truth table of an implication is counter-intuitive suggests that it should not have the status of a "definition," and is the subject of legitimate intellectual inquiry. Taking the long view, many historians and logicians would agree. The question becomes how can such a construction in propositional logic be presented from historical sources at a level for beginning undergraduates or students in secondary schools, particularly when there are so many topics in a chapter on logic, not to mention all the other topics in a textbook on discrete mathematics? How can anything be provided beyond the isolated factual information or "historical snippets" (Tzanakis and Arcavi, 2000) which are increasingly found in today's textbooks?

The curricular module *Deduction Through the Ages* responds to this question by offering a selection of excerpts from various primary sources, closely following a chronological order, designed to illuminate the final selections of Ludwig Wittgenstein (1889–1951) (Wittgenstein, 1921, 1922, 1961) and Emil Post (1897–1954) (Post, 1921), where the modern truth table of an implication appears. Certainly the module does not provide a complete history of propositional logic, nor does it prepare students to begin academic research in logic. It does, however, offer a more balanced and comprehensive overview of deductive thought than that offered by a purely formulaic treatment found in many textbooks. The module, covered in its entirety, could substitute for a chapter on logic, which has occurred in our teaching over the years 2008–2011.

Modern textbooks treat the truth value of $p \to q$ (p implies q) as a matter of settled logic, although in ancient Greece the truth of the hypothetical proposition "If a warrior is born at the rising of the Dog Star, then that warrior will not die at sea," was a matter of debate. The module begins with this debate and offers Philo of Megara's (ca. 4th century B.C.E.) verbal characterization of a valid hypothetical proposition as "that which does not begin with a truth and end with a falsehood" (Empiricus, 1967, II. 110). Students respond well to a verbal formulation of the issue, which initially follows the genetic principle of learning (Schubring, 2011), whereby the historical development of the subject parallels the learning process of the individual, although we do not claim that all historical sources in this module follow a genetic principle (nor was the module written with this principle in mind). The module does continue with the verbal statement of five argument forms attributed to the Greek philosopher Chrysippus (280–206 B.C.E.) (Diogenes, 1965) and raises the question for students and instructors whether these five rules could be special cases of just one rule, and, if not, which rules might be equivalent. For the reader's convenience, the rules are reproduced below (Diogenes, 1965, p. 189) (Gould, 1970, p. 83) (Long and Sedley, 1987, p. 212–213).

1. If the first, [then] the second. The first. Therefore, the second.
2. If the first, [then] the second. Not the second. Therefore, not the first.
3. Not both the first and the second. The first. Therefore, not the second.
4. Either the first or the second. The first. Therefore, not the second.
5. Either the first or the second. Not the first. Therefore, the second.

In the language of Sfard (2000, p. 161), Chrysippus's argument forms are *object-level* rules for deductive reasoning in ancient Greece. These rules, stated verbally, govern the content of discourse in how conclusions are drawn from hypotheses. The organizing principle (leitmotif) of the module is then a metadiscursive discussion of the possible equivalence of these individual discursive (object-level) rules. The tools for this analysis are the individual historical sources, chosen for their relevance to this issue. The individual discourses from the different historical selections are a source of commognitive conflict (Sfard, 2008), requiring for its resolution an understanding and "acceptance of the discursive ways of the expert interlocutor" (Sfard, 2008, p. 258), where the expert interlocutors are in this case the historical authors (and their contemporaries conversant in the authors' works). The leitmotif comes into view only via the juxtaposition of the sources throughout time, and is not apparent in any one source. Certainly Post is not building on Chrysippus, although the truth tables in Post's "Introduction to a General Theory of Elementary Propositions" (Post, 1921) can be used to settle the equivalence of Chrysippus's rules. Beginning an undergraduate course with Post's work, however, ignores the intellectual struggle that eventually found resolution in the computational technique of a truth table, not to mention how the precise meaning of language may have changed over time. Chrysippus's first rule begins "If the first, [then] the second," called an "implication" today. The other rules involve the terms "and," "or," and "not," which carry verbal meanings, ideal for in-class discussion, particularly whether "or" is meant to be inclusive, exclusive, or whether a distinction is being made. An examination of Chrysippus's rules requires then a study of these terms and further readings require us to confront how various historical authors have rendered the meanings of these terms.

To indicate how this is achieved, we briefly describe the other historical sources in this module, without reproducing all the excerpts which are contained in the text of the module. In the modern era, an initial attempt to render logic to a symbolic form is studied from George Boole's (1815–1864) treatise *An Investigation of the Laws of Thought* (Boole, 1854, 1958), where the terms "and," "or," and "not" are represented by the symbols $\times$, $+$, and $-$, respectively. Students are asked to compare the logical meaning of these symbols with their algebraic meaning. Since Boole introduces no symbol for an implication, it is difficult to discuss the equivalence of Chrysippus's first two rules with the others. Students do, however, witness certain laws of logic expressible entirely in symbolic form, such as for "literal symbols" (Boole, 1958) (true-false statements) x, y, z, we have, in Boole's notation,

$$x(y+z) = xy+xz,$$

meaning that "x and either y or z" is equivalent to "either (i) x and y or (ii) x and z." The logical content of Boole's work is encapsulated via the symbols $\times$, $+$, $-$, and the object-level rules for their use involve equations such as $x(y+z) = xy+xz$ above.

An entirely different system of notation, the "Begriffsschrift" (concept-script), is introduced from the work of Gottlob Frege (1848–1925), who sought a logical basis, not for language as Boole, but for arithmetic and mathematical analysis. In *The Basic Laws of Arithmetic* (Frege, 1962, 1964), Frege introduces the condition stroke

$$\begin{array}{l} \top\, \xi \\ \llcorner\, \zeta \end{array}$$

and declares that the entire symbol is false when the lower argument ζ is true and the upper argument ξ is false. Otherwise, the symbol is true. Thus, the condition stroke is true when it does not begin (ζ) with a true statement and end (ξ) with a false statement, which compares well with Philo's verbal statement that a valid hypothetical proposition is "that which does not begin with a truth and end with a falsehood." Frege does not introduce separate symbols for the logical connectives "and" and "or," but does include a symbol for "not" (negation) in his Begriffsschrift. With this in hand, four of Chrysippus's five argument forms can be written directly in terms of the condition stroke with variations on negating or affirming the stroke or its arguments ζ, ξ. Hence, four separate rules of discourse are replaced with one, an observation explored in the student exercises of the module. After a discussion of Frege's condition stroke and his symbol for a logical negation, for example, the module offers the following student exercise.

> Exercise: Using the "inclusive or," rewrite the example of Chrysippus's fifth rule of inference "Either it is day or it is night; but it is not night; therefore it is day" entirely in Frege's notation. Let ξ denote the proposition "It is day," and let ζ denote the proposition "It is night." Write the major premise, minor premise, and conclusion using the concept-script. Be sure to justify your answer using complete sentences.

These exercises engage the students in a metadiscursive analysis of the possible equivalence of Chrysippus's object-level rules. The goal of the Frege section is not the mastery of a new and strange system of notation, but the insight this notation offers into these equivalences. When asked to write summary papers about the curricular module, one student wrote: "The use of Frege notation really clarifies the relationship of the rules of logic and deductive thinking," which represents a metadiscursive observation.

Although Frege does not call his condition stroke an implication, it can be interpreted as such, but this is left as an insight to be developed after reading an excerpt from Bertrand Russell (1872–1970) and Alfred North Whitehead's (1861–1947) monumental *Principia Mathematica* (1910, 1912, 1913), where notation similar to that in use today is developed. These authors introduce separate symbols for "(1) The Contradictory Function, (2) the Logical Sum or Disjunctive Function, (3) the Logical Product, or Conjunctive Function, (4) the Implicative Function" (Russell and Whitehead, 1997, pp. 6–8). All of these have verbal precedents from ancient Greece and symbolic precedents from either Boole or Frege. Russell and Whitehead, however, offer an explicit definition of the implicative function (Russell and Whitehead, 1997, p. 6–8).

> The Implicative Function is a propositional function with two arguments p and q, and is the proposition that either not-p or q is true, that is, it is the proposition $\sim p \vee q$. Thus, if p is true, $\sim p$ is false, and accordingly the only alternative left by the proposition $\sim p \vee q$ is that q is true. In other words if p and $\sim p \vee q$ are both true, then q is true. In this sense the proposition $\sim p \vee q$ will be quoted as stating that p implies q. The idea contained in this propositional function is so important that it requires a symbolism which with direct simplicity represents the proposition . . . . The symbol employed for "p implies q", i.e. for "$\sim p \vee q$" is "$p \supset q$." This symbol may also be read "if p, then q." . . .

Here we see an implication defined as a certain (inclusive) “or” statement of equivalent deductive power, essentially that stated in Chrysippus’s fifth rule, and also equivalent to Frege’s condition stroke. To accept this as a definition of an implication requires a metadiscursive discussion (Sfard, 2000, p. 167) of the intent and goals of an implication, a discussion in which students are able to participate after an acquaintance with the historical background leading to the work of Russell and Whitehead. With this definition in hand, a metadiscursive discussion of the logical equivalence of Chrysippus’s rules is greatly facilitated, although not yet reduced to calculation.

In 1921 both Wittgenstein’s *Logische-philosophische Abhandlung* (Wittgenstein, 1921) and Post’s *Introduction of a General Theory of Elementary Propositions* (Post, 1921) appeared in print. Both authors introduce a tabular format for displaying the truth values of a proposition, and excerpts for their works are included in the module. Wittgenstein calls his diagrams “schemata,” while Post uses the term “truth table.” Post clearly displays the truth table for an implication in the notation of Russell and Whitehead. With these tables, a final computational argument concerning the equivalence of Chrysippus’s rules can be achieved.

In this curricular module we have seen various approaches to the object-level rules of deductive thought, from Chrysippus’s verbal argument forms to Boole’s arithmetical symbols to Frege’s concept-script to the definitions of Russell and Whitehead and their tabular displays in the work of Wittgenstein and Post. The struggle for a useful notation for these rules and a comparison of their use offers a commognitive conflict, which has found resolution in the presently accepted definitions of *Principia Mathematica* (Russell and Whitehead, 1910, 1912, 1913). By participating in a metadiscursive discussion across time, students gain a deeper appreciation of modern mathematics, and a sense of its purpose. Further student comments when asked to write a concluding paragraph about the module *Deduction Through the Ages*, read:

> “Someone who is studying the Russell and Whitehead notation can easily retrace and write the statement as an equivalent ‘if-then’ statement using one of Chrysippus’s rules. That student of logic can also be studying Frege and turn one of his condition strokes into a Post truth table. This interconnection makes knowing the history of logic and evolution of propositional logic useful if not absolutely necessary in being able to understand the logical concepts of today.”
>
> “With all of this newly acquired knowledge, I understand logic a lot better and my mind has been opened to a whole new view of sentences and their meanings.”
>
> “It is neat to see how each new generation of logicians built on previous work from former logicians, and how the work is all interconnected.”
>
> “The final mathematician that seems to effortlessly unite the verbal logic with the symbolic logic is Emil Post who developed the method we use now to represent the truth values of compound statements. His method of depicting propositional logic is a masterful display of how the development of mathematical logic has evolved.”
>
> “In conclusion, as shown above, logic has evolved in many different ways. From the days of Chrysippus all the way to Russell & Whitehead, logic has become what it is today. Before I started this class this past semester, I had never heard of

math being used in this way. It is interesting to know that logic had its bases in math this way."
"*Principia Mathematica* introduced the implicative function as a way of directly communicating an 'if-then' proposition. This goes to show that the concept of, and how to communicate 'if-then' statements has evolved over the millennia as mathematicians sought better ways of examining and approaching logic."
"Our project seemed involved in clarifying what implication is exactly, gaining effectiveness by abandoning a more technically formulated approach."
"With knowing the background of each concept, you then understand how to implement them in different situations. ... Some concepts had 'gaps' in them and were not useful in all situations, just most."
"The project does not say 'This is how it is'. The project says: This is how Aristotle saw it, this is how Chrysippus saw it, this is how Boole saw it, this is how Frege saw it, what do you think, you be the judge."

5 A Second Curricular Module Case Study: *Abstract awakenings in algebra*

The module described in the previous section illustrates how student engagement in the differing discourses of various historical authors through guided reading of excerpts focused on a specific mathematical topic can support student learning of that topic while simultaneously promoting insight into the nature of mathematics as an intellectual activity. Central to the design of such a module, both the juxtaposition of sources across different time periods and the incorporation of tasks which explicitly prompt students to reflect upon these sources at a metalevel emerge as key design issues. In this section, a different set of design issues are considered in the context of the extended module *Abstract awakenings in algebra* (Barnett, 2010).

Intended for use in a first course in abstract algebra, the original design plan for this project was to base it on just one original source, *On the theory of groups, as depending on the symbolic equation* $\theta^n = 1$. In this 1854 paper, Arthur Cayley (1821–1895) explicitly recognizes the common features of various (apparently) disparate mathematical developments of the early nineteenth century and defines a "group" to be any (finite) system of symbols subject to certain algebraic laws. Asserting (without proof) that the concept of a group corresponds to the "system of roots of [the] symbolic equation [$\theta^n = 1$]," Cayley states (without proof) several important group theorems and proceeds to classify all groups up to order seven. While focusing on the classification of arbitrary (finite) groups and their properties, Cayley does not neglect to motivate this abstraction through references to specific nineteenth century appearances of the group concept, including "the system of roots of the ordinary equation $x^n - 1 = 0$" and the theory of elliptic functions. He also remarks that "The idea of a group as applied to permutations or substitutions is due to Galois, and the introduction of it may be considered as marking an epoch in the progress of the theory of algebraical equations."

The fact that Cayley's paper provides such a powerful lens on the process and power of mathematical abstraction makes it simultaneously attractive and difficult to use in a classroom project aimed at developing an understanding of elementary group theory. In particular, his insight into the common features of a variety of existing mathematical objects was dependent upon his familiarity with those objects, and with existing discourses about them. The autopoietic nature which Sfard singles out as the distinguishing characteristic of mathematical discourse is thus clearly exemplified in this one source. Yet even at the time, Cayley's insight was premature and his paper attracted little attention from other mathematicians until late in the nineteenth century. Thus, simply reading Cayley's paper as one's first introduction to group theory — even within the framework of a guided reading module — seemed unlikely to lead to a robust understanding of that theory.

To identify prior source material that could lead up to a successful reading of Cayley's paper, the question "What did Cayley himself read?" was natural to consider. Among the mathematical works upon which Cayley was building, those of two authors emerge as especially important precursors of Cayley's definition of abstract group: the work of J. L. Lagrange (1736–1813) on algebraic solvability (Lagrange, 1770–1771, 1808) and that of Augustin Cauchy (1789–1857) on permutation theory (Cauchy, 1815a,b, 1844). The 90-page project resulting from this historical research develops a significant portion of the core elementary group theory topics from the standard curriculum of a first course in abstract algebra, and has been tested at three institutions as a textbook replacement for this part of the curriculum. To provide a sense of how the incorporation of pre-Cayley sources into this project provides scaffolding for student learning of the mathematical content itself while also properly framing the historical context of Cayley's work, we briefly summarize the primary sources used in each of the project's four major sections.

Following a brief introduction which surveys historical efforts to find solutions to general polynomial equations, the project's first major section is built around selections from pioneering work by J. L. Lagrange on the algebraic solution of polynomial equations. As with all his research, generality was Lagrange's primary goal in his works on equations. In seeking a general method of algebraically solving all polynomial equations, he began by looking for the common features of the solution methods for quadratics, cubics, and quartics. Following a detailed analysis of the known methods of solution, he concluded that one thing they had in common was the existence of an auxiliary (or *resolvent*) equation whose roots (if they could be found) would allow one to easily find the roots of the originally given equation. The project itself does not examine Lagrange's complete analysis, but draws instead on a summary of his findings in the 1808 edition of *Traité de la résolution des équations numériques* (Lagrange, 1808). The essential relationship between the roots of the given equation and the roots of an associated resolvent equation described in the following excerpt provides a springboard for the exploration of several group-related concepts throughout the project's first section (Lagrange, 1808, p. 295):

> In *Mémoire de l'Académie de Berlin* (1770, 1771), I examined and compared the principal methods known for solving algebraic equations, and I found that the methods all reduced, in the final analysis, to the use of a secondary equation called the *resolvent*, for which the roots are of the form
>
> $$x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{(iv)} + \dots$$

where $x', x'', x''', \ldots$ designate the roots of the proposed equation, and α designates one of the roots of unity, of the same degree as that of the equation.

Although this particular excerpt does not specify conditions on the value of α beyond that of being an m^{th} root of unity (i.e., a number for which $\alpha^m = 1$), Lagrange later makes it clear that α must be a "primitive complex-valued root of unity" (i.e., powers of α must generate all m distinct solutions of the equation $x^m - 1 = 0$). Excerpts from Lagrange's own detailed discussion of properties of such roots in Lagrange (1808) thus provide an authentic context for student exploration of concepts related to what are now called cyclic groups. Additionally, Lagrange's representation of the solutions to the equation $x^m - 1 = 0$ in terms of trigonometric functions (i.e., $x = \cos\frac{k}{m}360° + \sin\frac{k}{m}360°\sqrt{-1}$, where $k = 1, 2, 3, \ldots, m$) allow him to explore and justify certain properties of these roots through the geometric representation of trigonometric values on a unit circle. Students generally adopt a similar approach to complete a series of tasks related to Lagrange's observation that, for prime values of m, every complex m^{th} root of unity β except $\beta = 1$ is a primitive root. In these tasks, students first examine Lagrange's claim for several specific (prime and composite) values of m, before completing the details for a proof of the general claim based on an outline provided. Both the concrete mode of representation and the nature of Lagrange's discourse thus connect well with students' own "modern" experiences of polynomial equations and trigonometry, while preparing the ground for a move to the more abstract concepts (e.g., generators of cyclic groups) encountered later in the project.

The concept of a permutation also naturally arises in Lagrange's discussion of roots of unity, and again in his analysis of the relationship between roots of a polynomial and roots of its resolvents. With regard to the latter, for example, Lagrange argues that a m^{th} degree polynomial equation will have a resolvent degree $m!$ since "in the expression t, one can interchange the roots $x', x'', x''', \ldots, x^{(m)}$ at will since there is nothing to distinguish them here from one and another" so that "one obtains all the different values of t by making all possible permutations of the roots $x', x'', x''', \ldots, x^{(m)}$ and these values will necessarily be the roots of the resolvent in t which we wish to construct" (Lagrange, 1808, p. 297). The fact that permutations here constitute actions upon objects at the discursive level for Lagrange again connects well with the level of discourse of students at this early stage of the course. In contrast, Cauchy changes the objects of discourse by taking up the study of permutations as objects in their own right.

Cauchy's research on permutations was completed in two different periods, with two articles appearing in 1815 (Cauchy, 1815a,b), and the extensive *Mémoire sur les arrangements que l'on peut former avec des lettres données* and 27 other shorter articles appearing between 1844 and 1846. Although it is clear from his earliest works that Cauchy's ideas about permutations were initially stimulated by the specific problem of algebraic solvability (and Lagrange's work in that area in particular), Cauchy's own work again exemplifies the autopoietic nature of mathematics by focusing instead on the systematic development of the algebraic properties of permutations as interesting objects of study in their own right. Since many of the notational conventions and classification schemes for permutations currently in use today were first developed by Cauchy, reading his work offers students an authentic entry point into the study of permutation groups. For example, Cauchy's definition of the "order of a permutation P" as the least positive integer i for which $P^i = 1$ sets the stage for an analysis of the relation between the order of a permutation and the number of variables being permuted. Through this analysis, the equation $\theta^m = 1$, already encountered in the context of Lagrange's work on roots of unity, also enters into Cauchy's work on permutations, but now in the context of a non-commutative operation.

Cauchy's work on permutations further changes the objects of discourse by moving beyond operations on individual permutations to a study of "systems of conjugate permutations" (i.e., permutation groups) and their elementary theory, including a proof of what is now known as "Lagrange's Theorem" for groups of permutations (i.e., the order of any subgroup of a given finite group always divides the order of that group). The relatively concrete context of Cauchy's proof of this theorem allows students to develop an understanding of its meaning without becoming entangled in the abstraction of cosets, partitions and equivalence relations typically used to prove this result in modern textbooks. At the same time, the complete generalizability of Cauchy's proof to any finite group prepares students to make the transition to that level of abstraction later in the project. Project tasks requiring a careful examination of Cauchy's proof offer additional transitional support by prompting students to first implement his proof strategy within the context of a very specific subgroup example, and then to re-write the general proof using indexed notation to recursively define the array which forms the heart of that strategy. The instructions for this latter task also (implicitly) invite students to reflect upon Cauchy's proof at the metalevel by directing them to add detail and/or rephrase Cauchy's reasoning where they feel this is needed and/or helpful.

By unveiling important aspects of finite groups within the relatively concrete context of finite permutation groups, Cauchy's development of an independent theory of permutations thus serves as a bridge between Lagrange's earlier discourse on polynomial equations and Cayley's abstract theory of finite groups. As they work on the Cauchy section of the project, for example, students write comments such as:

> "It seems like this is related to the unit circle that Lagrange made use of."
> "I was thinking there had to be a connection between Cauchy's method & primitive roots of unity [in Lagrange]."
> "i-cycle and i^{th} root of unity 'behave' in a similar fashion. I'm sensing an abstract structure ..."

Note that all three of these quotes allude to the role played by the equation "$\theta^n = 1$" in the work of Cauchy and Lagrange, with the third quote also foreshadowing the change in discourse about this equation which occurs in Cayley's 1854 paper.

The last two sections of the project turn, finally, to a guided reading of Cayley's paper in its entirety. In section 3 of the project, definitions of group and subgroup, basic group properties (e.g., abelian, cyclic, uniqueness of identity and inverses, cancellation) and additional topics in elementary group theory (e.g., order properties of group elements) are extracted from Cayley's discussion of systems of symbols which satisfy the equation $\theta^n = 1$. Notably, Cayley begins by first describing the general properties (e.g., closure, associativity, an identity element) which the systems in which he is interested possess, and only later remarks that elements of such systems necessarily satisfy this equation. Also notable is the shift in Cayley's object-level discourse, with the focus now on systems of abstract symbols that satisfy certain properties, as opposed to any concrete instance of such a system (e.g., permutations, roots of unity). Consequently, the theorems which he claims can be deduced from those properties necessarily apply to every particular system which satisfies those properties.

Among these general theorems is the fact that every group of prime order is cyclic, an assertion which students both readily believe and can easily prove by drawing on their earlier work with primitive roots of unity in the Lagrange section of the project. Noting that this fact completely classifies all groups of prime order with this theorem, Cayley's paper then turns to the problem of classifying groups of composite order. Section 4 of the project explores the concept of group isomorphism through a careful reading of Cayley's proofs that there are only two groups of order 4 and only two groups of order 6. The representation of different groups by so-called "Cayley tables", which plays a central role at the object level in these proofs, is then further explored at the metalevel through various project tasks which prompt students to reflect upon what this representation reveals about the structure of a group. These reflections also serve as a means to promote recognition of the way in which Cayley tables implicitly suggest concepts in today's discourse about groups (e.g., cosets, quotient group).

One of the exciting aspects of implementing this project with students has been the way in which so many theorems of elementary group theory, as these are stated by Cayley, become obvious as a result of having read Lagrange and Cauchy in the first half of the project. Written comments made by students at this point in their work on the project include:

> "This is sort of the same thing we've been talking about over and over again."
> "Perhaps it's the slightly more modern language that makes Cayley's work easier to understand, or maybe I'm just more familiar with concepts after the previous sections."

This phenomenon was not wholly unexpected; after all, the rationale for including works from Lagrange and Cauchy in the project was to provide students with concrete exemplars (roots of unity and permutations respectively) of the abstract group concept identified by Cayley. What was not fully expected in advance was the extent to which those examplars prepared students for that shift in abstraction. Examining the project design through the lens of Sfard's theory of mathematical discourse offers insight into why this turns out to be the case.

Besides fleshing out the historical context of Cayley's work by including original source material from his predecessors, the project design also seeks to explicitly exploit the ways in which the mathematical context of today's students is different from that of Cayley to promote their understanding of group theory in its current form. For example, although Cayley's paper includes an example of a group consisting of operations on matrices, his pioneering work in matrix theory does not seem to have greatly influenced his thinking about groups. Today's students of abstract algebra, however, have typically completed a pre-requisite course in linear algebra which makes invertible matrices a natural example of a non-commutative group for them to consider. In terms of Sfard's work, matrices are already an object in student's discourse about mathematics which it seems natural to allow into this new discourse about groups. With respect to the theoretical frameworks discussed in section 3, however, this design decision clearly illustrates one aspect of the tension between a "historical way of knowing mathematics" and a "working mathematician's way of knowing mathematics" Fried (2007, p. 218) which mathematics educators encounter when bringing original sources to the classroom.

Other design decisions stemming from the use of this project as a replacement for such a substantial portion of a required course raise similar concerns. Ultimately, students need to know that they are leaving the course with a firm understanding of the current paradigm of group theory and an ability to read and write proofs that meet today's standard of rigor and formality. This concern led, for example, to a higher ratio of secondary commentary to primary source materials than is found in certain other projects, with this additional commentary including "modern" proofs of certain propositions, as well as examples (e.g., infinite groups) and terminology (e.g., isomorphism) that are not found in the original sources. In the final section of this paper, we briefly discuss how wrestling with these concerns has led to our own enriched understanding of the pedagogical work in which we are involved.

6 Conclusions and Caveats

In this paper, we have described a specific type of student project based on primary historical sources and designed to promote student learning of modern mathematics while simultaneously bringing that mathematics into relief against its historical development. We have also taken an initial step towards connecting these practical endeavors with theoretical frameworks which raise important issues about the nature of the work we are doing, while also offering insights into that work. In particular, we have examined two case studies from the perspective of Sfard's theory of thinking as communication (Sfard, 2008). We close by offering some final thoughts concerning the connections we see between our classroom materials and those frameworks.

Each of the modules described in our case studies offers guided readings of sequences of primary historical sources in mathematics, often involving quite different discourses, arranged around a common theme over different time periods. Interwoven with these sources are tasks that require students to actively engage with the mathematics of each source on its own terms, while also comparing it with those which came before and with present mathematical practice. Mathematics is thus to be viewed as a dynamic subject, whose signs and symbols have evolved over time and will continue to do so. If knowledge of this kind is part of the collective self knowledge of mathematics, then our curricular materials have a similar pedagogical goal as that stated in Fried (2007). These modules are written, however, with a culturally longitudinal theme in mind, not simply a curricular theme having as its goal teaching the current state of mathematics (Fried, in press).

Furthermore, consider our students' work in the context of the analyses of history in mathematics education described in section 3 which argue that for history to play an essential role — one which avoids the dangers of Whig history — requires not just primary sources, but also a hermeneutical approach to the reading of those sources. (See especially Jahnke et al (2002).) Within such a reading, the contrasts between past and present are to be consciously studied, always maintaining an awareness of oneself as the interpreter of the past. Only by embracing the natural tensions between past and present in this way, rather than whitewashing them, can a deeper understanding of both history and mathematics result. In our own projects, we now aim for both the sources we select and the student tasks we devise to achieve, among other things, a deep understanding of both the similarities and differences between past and present mathematics, not merely the past as a convenient or most natural avenue to the present. Yet even our materials are not immune to the dangers of a teleological reading in the hands of an instructor who is inexperienced in reading primary sources, or who has pedagogical goals different from our own. Instructors developing their own materials for teaching with primary sources are also cautioned against trying to find modern mathematics in any one historical source, and to avoid Whig history by adopting a hermeneutical approach in their materials.

Recasting this discussion in linguistical language from Saussure (1974) that has been studied by Fried (2007), one goal of a hermeneutical reading of primary sources is to understand the diachronic development of topics in modern mathematics, i.e., their development across time. Another of our overarching goals is to develop and improve students' ability to create and articulate mathematical arguments. In this regard, our projects require students to engage in activities that model how mathematicians work in a variety of ways: making sense of and interpreting information from various sources, making, testing, and refining conjectures, and describing results by referencing documentation, information, and intermediate results from testing conjectures. Because such activities are governed by metadiscursive rules which themselves vary between different mathematical discourses, the commognitive conflict which can arise when comparing sources addressing the same issue from different eras increases awareness of metadiscursive rules, in part by stimulating metalevel discussions concerning the formulation, intent and goals of underlying concepts (Sfard, 2008). Radical engagement with the disparate discourses of original sources selected from various mathematical communities appears to also support student learning by providing the scaffolding necessary to become a participant in a new (e.g., modern) mathematical discourse. This approach necessarily melds the use of history as both a tool and a goal (Jankvist, 2009).

Of course, the nature of questions we ask students to address influences not only their own engagement in metalevel analysis and hermeneutical interpretation, but also our own ability to document these phenomena in their learning. The nature of our project tasks, written by various authors over a number of years during which we were not consciously attempting to address metalevel learning, also varies considerably. Nonetheless, we have remarked that certain project tasks seem especially well suited to challenge students in these ways. As we have become more aware of their nature, importance, and efficacy, questions aimed at hermeneutic interpretation and metalevel analysis are precisely those which have been increasingly built into the "read, reflect, respond" tasks we write for our students. The two projects described as case studies in this paper are illustrative of the degree to which we have already shifted in this direction. Given the substantial intellectual rewards to be gained by students as they engage in such challenges, our current disposition is to incorporate even more such tasks into our projects. In this endeavor, the theoretical frameworks considered in this paper offer new challenges for us to consider, as well as new tools for analyzing the extent to which we are succeeding.

Our own practical experience over many years of undergraduate mathematics teaching, both with and without such projects, has already convinced each of us of the significant pedagogical value of implementing such projects in our own classrooms. Much of our evidence for this, however, lies in the good mathematical work which we have witnessed in our classrooms. With respect to the call of Jankvist and others for educational research which provides more than anecdotal evidence in support of the use of history in mathematics education, we note that our curricular modules fulfill, at least partially, "the 'urgent task' of developing critical implements for using history in the teaching and learning of mathematics" (Jankvist, 2009, p. 256). As evidence of their promise in achieving at least some of our goals for them, we close with a selection of responses from students when asked to comment on the benefits of learning from primary historical sources in post-course questionnaires in courses taught from historical modules:

"You get to learn not just the rules and theorems, but how and why they were developed."
"For me, being able to see how the thought processes were developed helps me understand how the actual application of those processes work. Textbooks are like inventions without instruction manuals."
"Helps me to link concepts and ideas more effectively than just being blindly told the result."
"To see the original not contaminated by intermediate interpretations is nice."
"A really big benefit is actually seeing what important mathematicians wrote, to see their material as they intended us to see it. I also like to see the contrast between what they claim is true and what is modernly true."
"You get to see how the people who generated these ideas thought and did their work."
"The original sources can be debated to form new interpretations."
"As a student you get to see where the math we do today came from and engage in the kind of thinking that was necessary to create it.
"It makes me care about learning."
"You get answers to questions like 'where did all this come from?' "
"I like being able to 'see' how the formulas were derived and what process got people in the past to their claims."
"You learn the concept."
"We learn directly from the source and attempt to learn concepts based off of the original proofs rather than interpretation of the original proof from someone else."

"I think historical sources help me understand the process of learning new techniques."
"It gives you the sense of how math was formed which prepares you for how to think up new, innovative mathematics for the future."

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