Quick, does 23/67 equal 33/97? A mathematician's secret from Euclid to today

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Abstract

How might one determine in practice whether or not 23/67 equals 33/97? Is there a quick alternative to cross-multiplying?

How about reducing? Cross-multiplying checks equality of products, whereas reducing is about the opposite, factoring and cancelling. Do these very different approaches to equality of fractions always reach the same conclusion? In fact they wouldn't, but for a critical prime-free property of the natural numbers more basic than, but essentially equivalent to, uniqueness of prime factorization.

This property has ancient though very recently upturned origins, and was key to number theory even through Euler's work. We contrast three prime-free arguments for the property, which remedy a method of Euclid, use similarities of circles, or follow a clever proof in the style of Euclid, as in Barry Mazur's essay [22].

1 Introduction

What may we learn if we ask someone whether 23/67 equals 33/97, and ask them to explain why? Is there anything subtle going on here? Our question is not about when two fractions should be *defined* as "equal," but rather about how in practice one is legitimately allowed to *determine* whether or not they are equal.

We first aim to stir up mud. We do this by asserting that there is a delicate issue in understanding fractions, something very powerful but sufficiently subtle that it tends early to become invisible to many mathematicians by assuming intuitive second nature. For others it remains entirely outside conscious awareness, despite perhaps being used occasionally in practice without question.

After stirring up mud, we will examine and compare three prime-free ways in which the issue can be resolved, from ancient to modern (via remedying Euclid using his algorithm, circular similarities, or following Mazur combining proportionalities). It is illuminating mathematically, historically via Euler, and pedagogically to see the full power of fractions harnessed without needing the standard modern appeal to uniqueness of prime factorization.

2 Equality of fractions

The issue of defining equality of fractions is nontrivial, since it is about conceptualizing the nature of proportionality, which is not nearly as simple as it sounds. Mathematicians today, however, are happy with equivalence classes of fractions creating the rational numbers. One forms the equivalence relation generated by $a/b = (ka)\,/\,(kb)$, and can verify that it is the same as the relation decreeing that $a/b = c/d$ if and only if $ad = bc$, after checking that this latter relation is transitive, which requires both commutativity and cancellation for natural numbers. This criterion, referred to as "cross-multiplying", sits well with the public too, especially if they are aware that it amounts to comparing the fractions via the specific common denominator bd.

We shall refer to converting a/b to $(ka)\,/\,(kb)$ as "expanding", and converting $(ka)\,/\,(kb)$ to a/b as "reducing". We have decreed that two fractions be equal precisely if they have a very specific identical expansion, which is easily checked to be equivalent to comparing them using any common denominator for expansion, for instance their least common denominator.

But what about reducing the fractions instead? As yet we have said nothing about what it may mean for two fractions to have or not to have an identical reduction. Does reduction just as definitively test fractions for equality, or not? Both computationally and theoretically it is often much preferable to reduce than to expand.

3 The mathematician's secret

Ask someone whether $23/67 = 33/97$. Expanding requires calculating two four-digit products, which some people will do, although they will certainly think twice before going to that much work. Some will more likely grab a calculator and convert to decimal quotients, even though this only yields approximations, so it might not answer the question. Some will try to determine whether one of the two fractions is greater by some comparison, even though this was not the question. Some will try reducing the fractions. If trying reduction, what will they conclude, and on what basis?[1]

The thoughtful mathematician has no need for multiplication, division, or size comparison. How so? What might the mathematician know about fractions that many others do not? The mathematician may choose neither to expand the fractions, nor to perform division, but rather to reduce them, and noticing quickly that $23/67$ is irreducible and that $33/97$ cannot reduce to it, asserts confidently that they cannot be equal. Voila! Surely this is incredibly powerful knowledge, but how obviously legitimate is it? After all, couldn't these two

[1] My unscientific surveys have shown that people, including mathematicians, display a remarkably diverse collection of knowledge and approaches to the question of determining equality. Surprisingly, even children from the same home and school may take quite different tacks, some expanding and some reducing. And to my astonishment, not all mathematicians even think of the possibility of reducing.

fractions still expand to something identical, even though they won't reduce to something identical? Can that really never happen? And who is aware of this?

All other questions about fractions can in principle be straightforwardly addressed. But the question of efficacy of reducing versus expanding fractions cannot easily be resolved: *Is it legitimate to use reduction of fractions to conclude that two fractions are unequal?*

By now the mathematician is perhaps slowly recalling the mantra that "every rational number is represented by a unique irreducible fraction" (called "lowest terms"), and that uniqueness is the key. And in the back of her or his mind, the modern mathematician, seduced always to reach for primes, imagines creating a proof using "uniqueness of prime factorization,"[2] and perhaps then starts wondering if this is truly necessary to justify quickly that $23/67 \neq 33/97$.[3]

To gauge how necessary unique factorization may be for reduction to be efficacious, let us step briefly inside the natural numbers and consider the nature of fractions using only certain categories of natural numbers forming acceptable multiplicative systems (specifically, commutative semigroups with cancellation):

1. The even integers. Here $2/6 = 6/18$, but both are irreducible (6 has no proper factorization in the even numbers), so the mathematician's method will not work: Different irreducible fractions can be equal.

2. The H-integers (after David Hilbert's use of them [5, 15]), consisting of the numbers one more than a multiple of three. Here $4/10 = 10/25$, again different irreducible fractions can be equal.

And in both cases cross-multiplying displays a failure of unique factorization.

We are thus appropriately admonished to recall that the natural numbers are "perfectly built," so that every fraction has a unique irreducible reduction. And indeed this powerful property of fractions is essentially unique factorization, even though primes are not mentioned. An amazing amount of beautiful and important mathematics hinges thereon.

In fact we should view it as almost miraculous that the natural numbers is a world where reduction is as effective as expansion, i.e., where reduction and expansion of fractions detect the same equivalence relation. After all, in any commutative semigroup with cancellation, although equality of fractions is precisely the equivalence relation "have an identical expansion", there is no reason why the relation "have an identical reduction" should even be transitive: For instance, in the H-integers above, $4/10$ and $40/100$ have an identical reduction, and so do $40/100$ and $10/25$, but $4/10$ and $10/25$ do not, so transitivity fails.

[2] I have oft wondered what intuition the average person has about whether prime factorization of natural numbers is unique or not, since it is so key to the nature of basic mathematics. Somewhat disappointingly, my informal surveys suggest that most people have no intuition one way or the other on this matter.

[3] Note that the questions of efficacy and efficiency are quite different. Reduction may be efficacious, always distinguishing unequal fractions, but whether it is more efficient than cross-multiplying is another matter. One might at first think that reduction is comparatively inefficient, requiring factorization before reduction, but in fact it does not, since the irreducible reduction of a fraction can be obtained by calculating the greatest common divisor of numerator and denominator using the Euclidean algorithm, which is highly efficient.

Returning to ordinary fractions, can we answer our reduction question without having first to introduce primes, then prove uniqueness of prime factorization, and then build another proof upon that? We ask: *Is there a straightforward way to deduce uniqueness of lowest terms for a rational number without invoking primes?*

4 Does reducing fractions tell all?

What we wish to know is that reducing ordinary fractions "tells all," i.e., that reduction can answer the equality question just as faithfully as expansion does when considering fractions based on the natural numbers. Certainly if two fractions have an identical reduction, they are equal. What we want to know is the converse.

Identical reduction of equal fractions: Given two equal fractions, there is some fraction they both reduce to.

This is easily seen to be equivalent to the following.

Unique irreducible reduction of fractions: Every fraction reduces to a unique irreducible fraction.

In turn, this is equivalent to the following two results.

Reduction tells all for fractions: Given two equal fractions, they have identical irreducible reductions.

Corollary (The mathematician's powerful secret): If two fractions have different irreducible reductions, they are not equal. Thus $23/67 \neq 33/97$.

Proof of equivalences. If we know "identical reduction of equal fractions," then a fraction can have only one irreducible reduction, since any other reduction would also have to reduce to it. If we know "unique irreducible reduction," then given two equal fractions, their irreducible reductions must be identical, since both are irreducible reductions of any identical expansion of the two equal fractions. Finally, if we know "reduction tells all," then given two equal fractions, they both reduce to their identical irreducible reduction. ∎

Thus proving any of these three equivalent properties of fractions will answer "Does reduction tell all?" affirmatively.

5 Euclid's VII.19, the four numbers theorem, unique factorization, and Euler

With the nub of *reduction tells all for fractions* recast as the property *identical reduction of equal fractions*, we find it in the historical literature from different points of view, as Euclid's *Elements* VII.19 in the language of Pythagorean proportionality [6, 23], and more recently called the four numbers theorem in the parlance of product decompositions [5, 16, 18, 26]:

- **Euclid, Book VII, Definition 20** (Pythagorean Proportionality, in modernized terminology): We say that $a : b = c : d$ if a is m n-th parts of b, and c is the same parts of d.

  By his "parts" language, Euclid means that there exist natural numbers x, y, m, n such that $a = mx$, $b = nx$, $c = my$, and $d = ny$. In the language of fractions, this simply says that a/b and c/d both reduce to m/n.

  Euclid, Proposition VII.19: $a : b = c : d$ if and only if $ad = bc$.

The "only if" is trivial, and the "if" has been called the four numbers theorem.

- **Four numbers theorem (**identical reduction of equal fractions): If $ad = bc$, then there exist x, y, m, n such that $a = mx$, $b = nx$, $c = my$, and $d = ny$.

  This was first named by László Kalmár as the Vierzahlensatz [16], and proven by him by mathematical induction. It appears that Kalmár did not notice that this is Euclid's VII.19.

In terms of divisibility, this is nicely recast as

- **Divisors of a product**: If d divides bc, then $d = ny$ where n divides b and y divides c.

Shifting our consideration only briefly to contrast with prime factorizations, from any of the above one can easily deduce what we often think of today as underpinning uniqueness of prime factorization.

Euclid's Lemma (Proposition VII.30) (modernized notation): If a prime p divides a product, then it divides one of the factors.

Proof from the viewpoint of fractions. If $p \mid ad$, then $ad = pl$, so $a/p = l/d$. Now either $p \mid a$, or else a/p is irreducible, in which case by *identical reduction of equal fractions* l/d reduces to a/p, and therefore $p \mid d$. ■

Proof from the viewpoint of divisors of a product. If $p \mid ad$, then by *divisors of a product* p is a product of two numbers, necessarily 1 and p, dividing a and d, and thus p itself divides either a or d. ■

Thus it should not surprise that many results proven today using unique factorization were at least as easily proven in the past without primes, e.g., directly from VII.19, also known as the four numbers theorem.

In fact, it is remarkable that uniqueness of prime factorization as we know it today was not formulated until barely two centuries ago in Carl Gauss's 1801 *Disquisitiones Arithmeticae.* Prior to Gauss, the focus was on knowing how to list all divisors of products of numbers. This is nicely accomplished via the four numbers theorem, which ensures that the collection of divisors of a product of two numbers comprises precisely, and crucially no more than, all the pairwise products of individual divisors of the two numbers. Of course from knowing all the divisors of a product we can easily obtain uniqueness of prime factorization, but the precise notion of unique factorization was neither explicitly considered by, nor of apparent interest or necessity to, those before Gauss. Mathematicians like al-Farīsī (c. 1267–1319) [2, 3], Jean Prestet (1648–1690) [12][24, Ch. 6, p. 146ff], and Leonhard Euler (1707–1783) [9, Book 1, Abschrift 1, §41] dealt very successfully with divisibility using results such as VII.19 and VII.30 [1, 4, 18, 20].

For instance, Euler made an occasional nod to Euclid as justification where today we appeal to unique factorization. Two illustrations from Euler's works [18, pp. 186–7], easily proven from the four numbers theorem, are his use of the following.

- If a product of two relatively prime numbers is a n-th power, then so are the individual factors.

  Euler explicitly says that thanks to Euclid it would be superfluous for him to provide a proof [7, Lemma 1, p. 126][18, p. 186][20, 21]. He used this result in proving Fermat's Last Theorem for exponent four [17].

- Any divisor of a product of two relatively prime numbers is uniquely a product of a divisor of each of them.

  This Euler needed in the development of his "φ function," which counts the natural numbers up to n that are relatively prime to n [10, Ch. 4, p. 16][20, 21], and in his study of amicable numbers, in order to know that the "sum of proper divisors of n" function $\sigma(n)$ is, for example, multiplicative on relatively prime products mn [8, Lemma 1, p. 24][10, Ch. 3][18, pp. 186–7][20]. And Euler at one point makes explicit use of the four numbers property [8, Prob. 2, p. 61][20].

In sum, prime factorizations are quite unnecessary for obtaining *reduction tells all* for fractions, and also for most of the number theory achieved by Euler; all that is needed is the four numbers theorem, Euclid's VII.19.[4]

Returning therefore to resolving the reduction question for fractions without mentioning primes, it appears at first blush that Euclid did this nicely for us with his VII.19, interpreted as *identical reduction of equal fractions*. However, we are about to see that there are very serious problems with this tidy conclusion.

6 Three prime-free proofs of "reduction tells all" for fractions

We look first to Euclid's VII.19, interpreted as *identical reduction of equal fractions*, for a prime-free approach to the equivalent *reduction tells all* for fractions. However, while Euclid's proof route is prime-free, it is incredibly long and circuitous, stretching over quite a sequence of propositions beginning with VII.5.

[4] Of course in many areas of modern algebra, unique factorization is crucial. The relationship between the four numbers theorem and unique factorization in the setting of ring theory is explored in [18, 19, 20].

And much worse, while we can streamline it quite a bit, it suffers from being wrong, containing a subtle but substantial gap.[5]

The first of the three prime-free proofs we will provide is inspired by our remedy[6] in [23] for Euclid's approach to VII.19. The second uses geometric circle similarities modeling modularity. And the third is a delightfully economical approach in the style of Euclid, as in Barry Mazur's [22], cleverly combining the equivalent proportional nature of different yet equal fractions. It is most illuminating to investigate where each of these proofs fails for the even integers and the H-integers discussed earlier.

Greatest common divisor for unique irreducible reduction

We prove *unique irreducible reduction* in a manner close to our fix [23] for Euclid's proof of VII.19.

Proof. Given a/b, let $g = \gcd(a, b)$ be the greatest common divisor of a and b, and write $a = ge$, $b = gf$, so e/f is an irreducible reduction of a/b. Now consider an arbitrary reduction c/d of a/b, so $a = hc$, $b = hd$.

We need a crucial fact about the natural numbers, that every common divisor of two numbers must divide their greatest common divisor (gcd). This is stated by Euclid as his Porism (corollary) to the Euclidean algorithm, and follows by examining the algorithm's iterative process both upwards and downwards.[7] It is an algebraic property of the gcd.

Thus h divides g, so writing $g = mh$ we have $a = mhe$ and $b = mhf$. Thus $c = me$ and $d = mf$, hence c/d reduces to e/f. Then, since every reduction of a/b reduces to e/f, the irreducible reduction is unique. ∎

In his proof of VII.19, Euclid assumed without justification something we discussed earlier, transitivity of the relation "have an identical reduction" on pairs of fractions, which involves dealing with multiple common divisors simultaneously.[8] We explained in [23] how Euclid could have justified this, using his gcd Porism in much the same way we just did above. We chose here to

[5]The serious gap in Euclid's proof of VII.19, not identified until 1970 [23, 27], leaves one wondering when Euclid's key number theoretic results, such as Euclid's Lemma (VII.30) relying on VII.19, were actually validly proven by anyone. Since al-Farīsī relied directly on Euclid's results such as VII.30 [3], it seems that Jean Prestet in 1689, who developed his results independently of Euclid, might have been the first [12]. Prestet uses the Euclidean algorithm to prove his key result, from which he derives results like VII.30 [24, Ch. 6, p. 146ff]. Interestingly, quite unlike Euclid, the key result Prestet first derives from a rather elaborate argument with the Euclidean algorithm is not about divisors of a product, but rather about least common multiples. He proves that the least number divisible by two relatively prime numbers is their product. All else follows from this.

[6]In [23] we analyzed Euclid's path to Euclid's Lemma (VII.30) via VII.19, including the major lacuna in his proof of VII.19, explained below after our first proof of unique irreducible reduction.

[7]Today we tend first to derive the much more recent Bezout equation, that $\gcd(a, b)$ is an integer linear combination of a and b, from the upwards process, and then Euclid's Porism therefrom. But Euclid obtains the Porism directly from the algorithm, without needing the Bezout equation.

[8]Euclid does not work with "fractions" per se, but his analysis involves the equivalent of this.

refocus Euclid's approach to prove *unique irreducible reduction* rather than the equivalent VII.19.

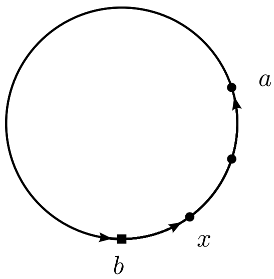
Circle similarity for the four numbers theorem

János Surányi [25, 26] gives a proof of the four numbers theorem using translation congruence properties of an integer lattice and similar triangles, that simultaneously automatically selects two gcds from the geometry. And Klaus Härtig and Surányi [5, 13, 14] make a geometric circle proof. It appears that Härtig and Surányi, like Kalmár, were unaware that the four number theorem is Euclid's VII.19. Of course they were also unaware that Euclid's proof is flawed.

Here we present a different geometric proof from theirs, with a modular flavor, using similarity of partitioning of circles, invented because I knew of the existence of a Härtig-Surányi circle proof, but couldn't immediately obtain a copy of it. To my surprise, the following proof is remarkably different from their circle proof, using geometric similarity of three circles rather than a counting analysis on a single circle.[9]

Four numbers theorem: If $ad = bc$, then there exist x, y, m, and n such that $a = mx$, $b = nx$, $c = my$, $d = ny$.
Proof. We may assume that $a < b$.

On a circle of circumference b, from the bottom lay down successive arcs of length a, marking their connecting endpoints. Since da is a multiple of b, the points will eventually repeat, so the circle is partitioned by the finitely many arcs joining adjacent points. Since the same partitioning must result when starting from any of the points, the arclengths between adjacent points are all the same, i.e., the points partition the circle into n arcs of equal length x, i.e., $b = nx$. Also a is a multiple of x, say $a = mx$. Moreover, x is just some multiple of the original increment a reduced by some multiple of b from the wrapping around the circle, i.e., $x = ra + sb$ for integers r, s. Thus x is itself an integer length.

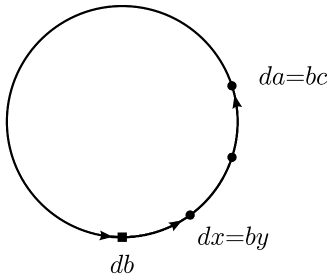


$$b = nx$$
$$a = mx$$
$$x = ra + sb$$

Now scale the partitioned circle up by a factor of d, to a similar circle with circumference bd and initial arclength $da(= bc)$, and arclength between adjacent points $dx = d\,(ra + sb) = rbc + sbd = b\,(rc + sd) = by$ with $y = rc + sd$ also an

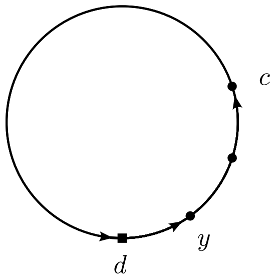
[9] The arguments do not really rely specifically on the geometry of circles, and could equally be phrased with polygons.

integer.



$$\begin{aligned} dx &= d\,(ra + sb) \\ &= bcr + bsd \\ &= b\,(rc + sd) \\ &= by \end{aligned}$$

Then scale this circle down by a factor of b, yielding a circle with circumference d, initial arclength c, and arclength y between adjacent points.



$$\begin{aligned} d &= ny \\ c &= my \end{aligned}$$

The two geometric similarities have rescaled the lengths of arcs, converting b to d, a to c, and x to y, while the multiples encoding the relationships between arcs remain the same, so the original $a = mx$ and $b = nx$ rescale to $c = my$ and $d = ny$. ■

It is geometrically obvious that $x = \gcd(a, b)$. Or, note that since $x \mid a, b$ and $x = ra + sb$ (the Bezout equation), we have $x = \gcd(a, b)$. Likewise $y = \gcd(c, d)$.

The rotational symmetry relied on in combination with similarity leads directly to both gcds occurring simultaneously with the same multiplicities, thereby avoiding needing Euclid's Porism on divisibility properties of greatest common divisors.

Combining equivalent proportionalities for unique irreducible reduction

Finally, we present a proof of *unique irreducible reduction* following Barry Mazur's recent essay [22][10], in which he studies and remedies Euclid's Proposition VII.20 in combination with VII.19 to prove Euclid's Lemma[11]. His proof hones in on just what is needed for unique irreducible reduction in a fashion that avoids entanglement with the lacuna in Euclid's proof of VII.19.

Proof (as in B. Mazur's essay [22]). Given a/b, let e/f be the fraction with smallest numerator that is equal to a/b, i.e., for which $af = be$. Our goal

[10] For a variant proof, see also [11, 20].

[11] In essence VII.19 and VII.20 together prove unique irreducible reduction.

is to show that a is a multiple ke, for then it will follow that $b = kf$, so a/b must reduce to e/f. Then e/f must be the unique irreducible fraction equal to a/b, since all fractions equal to a/b will then reduce to the one with smallest numerator.

If $a = e$, we are done. If not, let ne be the largest multiple of e still less than a. Then one checks that also nf will be less than b. Now verify the equality $(a - ne) / (b - nf) = e/f$ by cross-multiplying.[12] Then since e/f has smallest numerator among all fractions equal to it, we must have $a - ne \geq e$, in addition to $a - ne \leq e$ from the choice of n. Hence $a - ne = e$, so a is a multiple of e as desired. ■

The proof that Mazur gives of *unique irreducible reduction* requires no primes, no Euclidean algorithm, no algebraic divisibility of greatest common divisor, no geometry, no mathematical induction, and no contrapositive or contradiction. It relies on so little, i.e., repeated but bounded subtraction of smaller from larger, and a unit to allow repeated addition to transform to integer multiples[13]. Its simplicity is enabled by the clever choice of new third equal fraction based on the given different but equal ones.

7 What to conclude?

We have examined three prime-free ways of showing that *reduction tells all for fractions* (the mathematician's secret), equivalent to *unique irreducible reduction* of fractions and to *identical reduction of equal fractions*:

- À la Euclid (repaired via Euclidean algorithm),
- Geometric circle similarity (modularity),
- Mazur's [22] (mixing equal proportions).

The beautiful proof that Mazur gives of unique irreducible reduction, based solely on a single clever combination of proportionalities, is remarkably simple and satisfying, and one Euclid could have been very happy with. Is it perhaps also the shortest and most elementary path to uniqueness of prime factorization, since Euclid's Lemma follows immediately?

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[12] This equality, a somewhat bizarre move to us today, is precisely the sort of proportionality phenomenon that was second nature to the ancients.

[13] These are the two features missing, respectively, from the H-integers and the even integers in the examples of failure given earlier.

References

[1] A. G. Ağargün, The fundamental theorem of arithmetic dissected, *Math. Gaz.* **81** (1997) 53–57.

[2] —, Kamal al-Din al-Farīsī and the fundamental theorem of arithmetic, in *Science in Islamic civilisation* (Istanbul, 1991/1994), Stud. Sources Hist. Sci. Ser. 9, Res. Cent. Islam. Hist. Art Cult. (IRCICA), Istanbul, 2000. 185–192.

[3] A. G. Ağargün, C.R. Fletcher, al-Farīsī and the fundamental theorem of arithmetic, *Hist. Math.* **21** (1994) 162–173.

[4] A. G. Ağargün, E.M. Özkan, A historical survey of the fundamental theorem of arithmetic, *Hist. Math.* **28** (2001) 207–214.

[5] P. Erdős, J. Surányi, *Topics in the theory of numbers.* Springer, New York, 2003.

[6] Euclid, *The thirteen books of Euclid's elements.* Translated by T. L. Heath. Dover, New York, 1956, available at `aleph0.clarku.edu/~djoyce/java/elements/Euclid.html`.

[7] L. Euler, Theorematum quorumdam arithmeticorum demonstrationes, E98, *Comm. Acad. Sci. Petrop.* **10** (1738/1747) 125–146, available at `http://www.eulerarchive.org/`.

[8] —, De numeris amicabilibus, E152, *Opusc. Varii Argumenti* **2** (1750) 23–107, available at `http://www.eulerarchive.org/`.

[9] —, *Vollstandige Anleitung zur Algebra* (Petersburg 1770), E387, Reclam, Leipzig, 1920, available at `http://www.eulerarchive.org/`.

[10] —, Tractatus de numerorum doctrina capita sedecim quae supersunt, E792, *Comment. Arithm.* **2** (1849) 503–575, available at `http://www.eulerarchive.org/`.

[11] A. Genocchi, Démonstration de quelques propositions d'arithmetique d'après Euclide, *Nouv. Ann. Math.* **13** (1854) 426–428, available at `http://www.numdam.org/item?id=NAM_1854_1_13__426_0`.

[12] C. Goldstein, On a seventeenth century version of the "fundamental theorem of arithmetic", *Hist. Math.* **19** (1992) 177–187.

[13] K. Härtig, J. Surányi, Combinatorial and geometric investigations in elementary number theory, *Mat. Lapok* **23** (1972) 25–29. (Hungarian)

[14] —, Combinatorial and geometric investigations in elementary number theory, *Periodica Mathematica Hungarica* **6** no. 3 (1975) 235–240.

[15] D. Hilbert, *Zahlentheorie*, Lecture Notes 1897–1898. Edited by E. Maus. Univ. Göttingen, Göttingen, 1990.

[16] L. Kalmár, A számelmélet alaptételéről (Über den Fundamentalsatz der Zahlentheorie), *Mat. Fiz. Lapok* **43** (1936) 27–45. (Hungarian; German summary)

[17] R. Laubenbacher, D. Pengelley, *Mathematical expeditions: Chronicles by the explorers*, Springer, New York, 1999.

[18] F. Lemmermeyer, Zur Zahlentheorie der Griechen: Teil I. Euklids Fundamentalsatz der Arithmetik, *Math. Semesterber.* **55** (2008) 181–195, available at http://www.rzuser.uni-heidelberg.de/~hb3/publ/euk-1.pdf.

[19] —, Zur Zahlentheorie der Griechen: Teil II. Gaußsche Lemmas und Rieszsche Ringe, *Math. Semesterber.* **56** (2009) 39–51.

[20] —, *Unique factorization: The fundamental theorem of arithmetic*, book in progress.

[21] —, *The quadratic reciprocity law: With an appendix by Oswald Baumgart*, book in progress.

[22] B. Mazur, How did Theaetetus prove his theorem, in *The envisioned life: Essays in honor of Eva Brann,* Edited by P. Kalkavage and E. Salem, Paul Dry Books, Philadelphia, 2007. pp. 235–253, available at http://www.math.harvard.edu/~mazur/preprints/Eva.pdf.

[23] D. Pengelley, F. Richman, Did Euclid need the Euclidean algorithm to prove unique factorization?, *American Mathematical Monthly* **113** (2006) 196–205, available at http://www.math.nmsu.edu/~davidp/euclid.pdf.

[24] J. Prestet, *Nouveaux elemens des mathematiques ou principes generaux de toutes les sciences qui ont les grandeurs pour objet*, second edition, Vol. 1. Pralard, Paris, 1689.

[25] J. Surányi, Some proofs about the uniqueness of the prime number factorization of the integers, *Matematikai Lapok* **27** (1976) 213–226. (Hungarian)

[26] —, Schon die alten Griechen haben das gewusst, in *Grosse Augenblicke aus der Geschichte der Mathematik,* Edited by R. Freud, Akadémiai Kiadó, Budapest, and B. I. Wissenschaftsverlag, Mannheim, 1990. 9–50.

[27] C. M. Taisbak, *Division and Logos. A Theory of Equivalent Couples and Sets of Integers*, Odense University Press, Odense, 1971.

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