TEACHING AND LEARNING MATHEMATICS FROM PRIMARY HISTORICAL SOURCES

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Abstract: Why would anyone think of teaching and learning mathematics directly from primary historical sources? We aim to answer this question while sharing our own experiences, and those of our students across several decades. We will first describe the evolution of our motivation for teaching with primary sources, and our current view of the advantages and challenges of a pedagogy based on teaching with primary sources. We then present three lower-division case studies based on our classroom experience of teaching discrete mathematics courses with student projects based on primary sources, and comment on how these could be adapted for use with other lower division audiences.

1 INTRODUCTION

Why would anyone think of teaching and learning mathematics directly from primary historical sources? We aim to answer this question while sharing our own experiences, and those of our students across several decades. Our contention is that incredible richness and enhancement of student learning and appreciation can emerge from studying primary sources. We have several models to share, ranging from short student projects to entire courses built around primary sources, and at all undergraduate (and some graduate) levels.

In this article, we focus on single-topic projects that are suitable for use in lower division courses. We first describe the evolution of our motivations for teaching with primary sources, and current view of the advantages and challenges of a pedagogy of teaching with primary sources. We explain how we were first inspired, for which courses and levels we have experience, how we collaborated, what materials we found, used, and created, how we teach this way, and what the results are for

instructors and students. We then present three case studies based on our classroom experiences teaching with primary sources in introductory discrete mathematics courses. In addition to describing the design and implementation of these three student projects, we explain how each can be adapted for use with other lower division audiences (e.g., liberal studies majors, prospective elementary teachers and first-year calculus students).

2   INSPIRATION, EVOLUTION, BENEFITS, AND CHALLENGES

Our fluid collaboration was first catalyzed by William Dunham's 1986 article [13] about a historical "great theorems" course based on secondary material he prepared for students. Desiring both to imitate and expand, we gave students in honors courses primary sources on great topics in the development of mathematics. Our experience since is that the incredibly valuable intellectual questions and issues so often raised by carefully chosen primary sources could not be recreated by writing secondary materials, even if one tried; only the real, authentic original source has the innate pedagogical richness.

Our initial "great theorems" courses led to two books at the lower [21] and upper [17] division, containing annotated sequences of primary sources covering nine great mathematical themes. Each theme follows a grand mathematical story, built as a unit surrounding annotated primary sources, with exercises for students. During these early years of our students learning from primary sources we perceived an initial collection of benefits, discussed in [18, 19, 20], but as our own experience has evolved, we now see these initially perceived benefits falling in a somewhat passive "motivate, see, witness, experience" category.

Around 2003 our team enlarged and aimed to incorporate primary sources into regular required courses for mathematics and computer science majors. We have developed primary source materials for core content of courses on discrete mathematics, abstract algebra, algorithms, automata and formal languages, calculus, combinatorics, data structures, graph theory, mathematical logic, programming languages, and theory of computation. Our chosen pedagogical vessel for these sources has been the "student project," a guided reading module designed for easy insertion in a course, often replacing a standard textbook treatment of a particular topic. Each module is limited in scope for flexible use, often focusing on one or very few carefully selected primary source excerpts. Our now over 30 project modules and many details of the various ways we teach with them are available in [5, 6, 8, 10].

The format for each reading module carefully guides students to follow the evolution of mathematical concepts via original source excerpts.

The module contains discussion of historical context, biography of source authors, and mathematical significance and interrelations of the selections, creating a coherent pedagogical whole. Each module also contains detailed "Notes to the instructor" with suggestions of options, and indications of what they may expect to encounter with students. A key feature of each project is a series of exercise tasks for students to accomplish as they work through the module, intended to provoke students to develop their own understanding of the mathematical content based on the primary sources as stimuli. Some tasks ask students to fill in missing proof details, or to reflect explicitly on the nature of the mathematical process by answering questions about the level of rigor and proof style exhibited in the excerpt. Our students thereby progress naturally and consciously in their ability to construct proofs meeting today's standards. We can also introduce our students to present-day notation, terminology, and definitions as an outgrowth of studying the original texts; these devices emerge naturally as useful for modern sense-making of the primary sources.

Note that these latter features of the reading modules move beyond the "motivate, see, witness, experience" benefits mentioned earlier. Thus in [7] we elucidated further goals that now both guide our conscious design and constitute further support for "why" we teach with primary sources.

Thus we are using primary sources not only to introduce mathematics via authentic motivation, but as challenging texts for students actively to "interpret" as a component of creating their own meaningful understanding of modern mathematics. Our tasks for students now therefore often incorporate a more active "read, reflect, respond" paradigm. In [9] we examine our goals and methods in the context of modern educational frameworks on the role of history in mathematics education. This includes exploring the distinction between "history-as-a-tool", which includes the majority of the goals mentioned above, and "history-as-a-goal"[16].

There is no question that teaching with primary sources is a challenge requiring appropriate materials and an interest in exploring, learning, and teaching the full richness of mathematics through its history and the primary sources thereof. The ensuing intellectual rewards and satisfaction for us and our students have been so immeasurably large as to make it a worthwhile journey. Regarding materials, the ready availability of primary sources, in both their raw form and packaged for teaching, has grown so much in the past few decades that the truly interested instructor will readily find prepared materials or be able to create her own. An attempt at a compendium of resources for starting the journey is at [25]. The case studies which follow describe three of our own contributions to this compendium.

3   THREE CASE STUDIES TEACHING LOWER DIVISION COURSES WITH PRIMARY SOURCES

### 3.1   Hamilton Circuits and the Icosian Game

Topics from graph theory are today treated in a variety of courses, ranging from general education courses for liberal arts majors to undergraduate courses in discrete mathematics to specialized courses at the graduate level. Yet despite its current status as a powerful mathematical theory with applications in the physical, biological, and social sciences, the motivating questions behind the initial development of graph theory sometimes appear little more than interesting puzzles or games. In the instance of the 'Icosian Game', developed by Sir William Rowan Hamilton (1805–1865), this observation seems quite literally true.

In fact, the wholesale dealer in games *John Jacques and Sons* purchased the rights to Hamilton's idea for 25 pounds and marketed it as a board game in 1859. A second three dimensional version of the game marketed under the commercial name 'A Voyage Round the World' by the same company required players to visit twenty important locations such as Brussels and Zanzibar. Although both games were apparently too easy to make for a commercial success, the Icosian Game encapsulates a number of important mathematical ideas which are explored in the original source based project "Early Writings on Graph Theory: Hamiltonian Circuits and The Icosian Game" [3].

Initially developed for use in an introductory undergraduate course in discrete mathematics, the full version of this project begins with a brief historical introduction, followed by two separate mathematical sections. The first of these sections, "The Icosian Game and Hamiltonian Circuits," is also ideally suited for use in a liberal studies mathematics course. Assuming no prior knowledge of graph theory, this section of the project begins with the preface to the instructions pamphlet prepared by Hamilton for *John Jacques and Sons* [11, pp. 32–35]:

<center>∞∞∞∞∞∞∞∞∞∞</center>
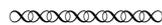
In this new Game (invented by Sir WILLIAM ROWAN HAMILTON, LL.D., &c., of Dublin, and by him named *Icosian* from a Greek word signifying 'twenty') a player is to place the whole or part of a set of twenty numbered pieces or men upon the points or in the holes of a board, represented by the diagram above drawn, in such a manner as always to proceed *along the lines* of the figure, and also to fulfill certain *other* conditions, which may in various ways be assigned by another player. Ingenuity and skill may thus be exercised in *proposing* as well as in *resolving* problems of the game. For example, the first of the two players may place the first five pieces
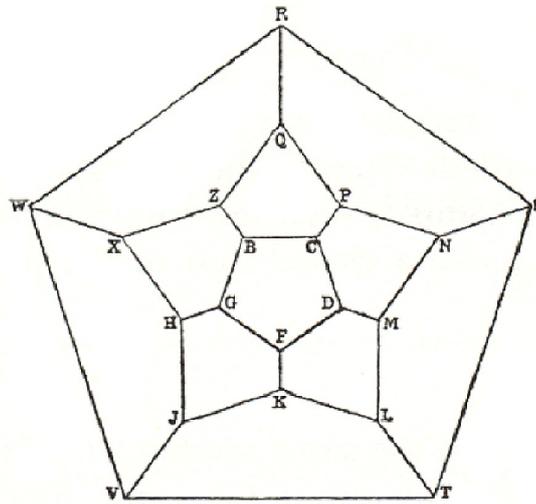
in any five consecutive holes, and then require the second player to place the remaining fifteen men consecutively in such a manner that the succession may be *cyclical*, that is, so that No. 20 may be adjacent to No. 1; and it is always possible to answer any question of this kind. Thus, if B C D F G be the five given initial points, it is allowed to complete the succession by following the alphabetical order of the twenty consonants, as suggested by the diagram itself; but after placing the piece No. 6 in hole H, as before, it is *also* allowed (by the supposed conditions) to put No. 7 in X instead of J, and then to conclude with the succession, W R S T V J K L M N P Q Z.

∞◊∞◊∞◊∞◊∞◊∞
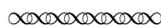
Following this preface, Hamilton provided several examples of Icosian Problems and their solutions. Project tasks prompt students to work through two of these examples in order to familiarize themselves with the concepts of 'Hamiltonian cycle' and 'Hamiltonian path', with both terms introduced in the project narrative along with other graph theory vocabulary (e.g., vertex, edge). The project also takes advantage of the very concrete context presented by the game as a means to develop students' ability to construct proofs. In the following exercise task, for instance, students make use of color pencils on a copy of the Icosian Game diagram to verify a particular claim made by Hamilton.

∞◊∞◊∞◊∞◊∞◊∞
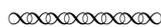
Exercise 4. In *Example 3*, Hamilton specifies $B\,C\,P\,N\,M$ as the first five vertices in the desired circuit. He then claims that the two solutions listed in the example are the *only* two solutions of this particular problem. Prove that these are in fact the only two solutions by completing the details of the following argument. Include copies of the diagram illustrating each step of the argument in a different color as part of your proof.

1. Explain why the initial conditions for this example imply that the solution to the problem must include either the sequence $R\,S\,T$ or the sequence $T\,S\,R$.
2. Explain why the initial conditions for this example imply that the solution to the problem must include either the sequence $R\,Q\,Z$ or the sequence $Z\,Q\,R$.
3. Explain why we can now conclude that the solution to this problem must include either the sequence $X\,W\,V$ or the sequence $V\,W\,X$.
4. Explain why the initial conditions for this example imply that the solution to the problem must include either the sequence $F\,D\,M$ or the sequence $M\,D\,F$.
5. Explain why we can now conclude that the solution to this problem must include either the sequence $K\,L\,T$ or the sequence $T\,L\,K$.
6. Use the information from above concerning which edges and vertices we know must be part of the solution to prove that the two circuits Hamilton lists are the only two solutions to the problem.

<div align="center">⚭⚭⚭⚭⚭⚭⚭⚭⚭</div>
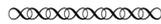
Excerpts from Hamilton's more scholarly articles about the Icosian Game idea were also included in the game's instruction pamphlet, and are used in the second mathematical section of the project to establish a link between the game and a specific example of a non-commutative algebra. In fact, Hamilton developed the game explicitly to provide yet another example of a non-commutative algebraic system, following his breakthrough discovery of quaternions in 1843. The Icosian Game system consists of three symbols, ι, κ, λ, subject to the following (non-commutative) rules:

$$\iota^2 = 1\,,\ \kappa^3 = 1\,,\ \lambda^5 = 1\,,\ \lambda = \iota\kappa\,,\ \iota\kappa \neq \kappa\iota.$$

Hamilton also described how to interpret this algebraic system geometrically within the Icosian Game. For instance, the operation ι reverses

a line of the figure (e.g., changing, B C to C B), while the operation λ changes a line considered as a side of a pentagon to the following side of that pentagon, adopting a right-hand rule for every pentagon except the outer one (e.g., changing B C to C D, but S R to R W). Through a series of exercises in this portion of the project, students independently explore both the symbolic algebra and its geometric interpretation.

Because no prior background in graph theory is assumed in the project, the connection to symbolic algebra makes the full project suitable for use in a junior-level abstract algebra course, as well as an introductory course in discrete mathematics. In either course, the entire project may be completed by students working in small groups over 2–3 in-class days, or assigned as a week-long individual project outside of class. For a liberal studies course in which only the first section of the project is completed, 1–2 days of in-class group work would suffice, with the world travel theme of the three-dimensional version of the game providing a natural segue from the project into a discussion of the Traveling Salesman Problem. Several other projects in our collection use original sources to explore other aspects of modern graph theory [2, 4].

## 3.2 Binary Arithmetic in Discrete Mathematics and General Education

We next outline two historical projects that highlight the development of binary arithmetic and its use in electronic digital computation today. The projects are at an elementary level and may be used in either a beginning discrete mathematics course, a general education course, or an introductory computer science course. Sections of both projects could also be used in courses for prospective elementary teachers as a means to explore and reinforce the place value concept, and to provide a broader cultural perspective about the practice of mathematics to these future teachers. For college courses, each project requires about two weeks for completion, although shorter excerpts of each project could be adapted, which require less time.

The first project "Binary Arithmetic from Leibniz to von Neumann" [10, 23] begins with an excerpt from Gottfried Wilhelm Leibniz's [1646–1716] "Explication de l'arithmétique binaire, qui se sert des seuls caractères 0 et 1, avec des remarques sur son utilité, et sur ce qu'elle donne le sens des anciennes figures Chinoises de Fohy" (An Explanation of Binary Arithmetic Using only the Characters 0 and 1, with Remarks about its Utility and the Meaning it Gives to the Ancient Chinese Figures of Fuxi) [14, p. 223–227], [22], in which Leibniz cites order, harmony and ease of calculation as advantages of binary numeration. He also cites a possible common root of the *Yijing* and Christianity with 0 denoting nothing and 1 denoting God [30]. Concerning binary arithmetic, the use

of only two digits eliminates the need to memorize a base ten multiplication table as well as methods of trial and error for long division, all of which are advantages of the binary system. Leibniz also claims that a monetary system with coins (or bills) in currency denominations of 1, 2, 4, 8, ... , units would be more efficient, and he cites a similar economy of scale to devise a system of weighing masses on a balance. The project opens with the following exercise about a two-pan balance.

∞∞∞∞∞∞∞∞∞
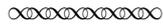
Exercise 1. Suppose that a two-pan balance is used for weighing stones. A stone of unknown (integral) weight is placed on the left pan, while standard weights are placed only on the right pan until both sides balance. For example, if standard weights of 1, 4, 6 are used, then a stone of weight 7 on the left pan would balance the standard weights 1 and 6 on the right. Two standard weights with the same value cannot be used. Leibniz realizes that all stones of integral weight between 1 and 15 inclusive can be weighed with just four standard weights. What are these four standard weights? Explain how each stone of weight between 1 and 15 inclusive can be weighed with the four standard weights. Make a table with one column for each of the four standard weights and another column for the stone of unknown weight. For each of the 15 stones, place an "X" in the columns for the standard weights used to weigh the stone.

∞∞∞∞∞∞∞∞∞
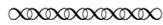
After completing the above exercise, students begin reading the excerpt from Leibniz, interspersed with additional exercises.
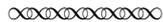
∞∞∞∞∞∞∞∞∞

An Explanation of Binary Arithmetic Using only the Characters 0 and 1, with Remarks about its Utility and the Meaning it Gives to the Ancient Chinese Figures of Fuxi

Ordinary arithmetical calculations are performed according to a progression of tens. We use ten characters, which are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, that signify zero, one and the following numbers up to nine, inclusive. After reaching ten, we begin again, writing *ten* with 10, and ten times ten or *one hundred* with 100, and ten times one hundred or *one thousand* with 1000, and ten times one thousand with 10000, and so on.

Exercise 2. Write the numbers 1, 10, 100, 1000 and 10000 as powers of ten. Express your answer in complete sentences or with equations. What pattern do you notice in the exponents?

But instead of the progression by tens, I have already used for several years the simplest of all progressions, that by twos, having found that this contributes to the perfection of the science of numbers. Thus I use no characters other than 0 and 1, and then, reaching two, I begin again. This is why *two* is written here as 10, and two times two or *four* as 100, and two times four or *eight* as 1000, and two times eight or *sixteen* as 1000, and so on.

Exercise 3. Write the numbers 1, 2, 4, 8 and 16 as powers of two. Express your answer in complete sentences or with equations. What pattern do you notice in the exponents? How do the exponents compare with those in Exercise 2? How does the progression by twos compare with the standard weights in Exercise 1?

Here is the *Table of Numbers*[1] according to this pattern, which we can continue as far as we wish.

Exercise 4. Compare the entries from 1 to 15 in Leibniz's "Table of Numbers" with the table for weighing stones that you constructed in Exercise 1. ...

<center>∞∞∞∞∞∞∞∞∞∞</center>
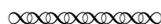
The project continues with arithmetical examples of addition, subtraction, multiplication and long division taken from Leibniz's paper, all done in base two. Students are asked to appreciate Leibniz's claimed advantages for multiplication and division when these operations are performed using the binary system (memorization of multiplication tables is not needed, and trial and error is avoided for long division).

The project then examines John von Neumann's (1903–1957) outline for one of the first programmable computers known as the Electronic Discrete Variable Automatic Computer (EDVAC) [31], written in 1945. For arithmetical operations, von Neumann cites nearly identical advantages with a binary system as Leibniz. When numbers are stored electronically via their digits, von Neumann recognizes a key advantage to the base two system. He writes [31]:

<center>∞∞∞∞∞∞∞∞∞∞</center>

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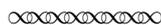
[1]The table, not shown here due to length, contains the numbers from one to 32 inclusive and their base two (binary) expressions.

First Draft of a Report on the EDVAC

2.2 First: Since the device is primarily a computer, it will have to perform the elementary operations of arithmetic most frequently. There are addition, subtraction, multiplication and division: $+$, $-$, $\times$, $\div$. ... [A] *central arithmetical* part of the device will probably have to exist, and this constitutes *the first specific part: CA*.

4.3 It is clear that a very high speed computing device should ideally have vacuum tube elements. Vacuum tube aggregates like counters and scalers have been used and found reliable at reaction times (synaptic delays) as short as a microsecond ($= 10^{-6}$ seconds), ...

5.1 Let us now consider certain functions of the first specific part: the central arithmetical part CA.

The element in the sense of 4.3, the vacuum tube used as a current valve or *gate*, is an all-or-none device, or at least it approximates one: According to whether the grid bias is above or below cut-off; it will pass current or not. It is true that it needs definite potentials on all its electrodes in order to maintain either state, but there are combinations of vacuum tubes which have perfect equilibria: Several states in each of which the combination can exist indefinitely, without any outside support, while appropriate outside stimuli (electric pulses) will transfer it from one equilibrium into another. These are the so called *trigger circuits*, the basic one having two equilibria. ... The trigger circuits with more than two equilibria are disproportionately more involved.

Thus, whether the tubes are used as gates or as triggers, the all-or-none, two equilibrium arrangements are the simplest ones. Since these tube arrangements are to handle numbers by means of their digits, it is natural to use a system of arithmetic in which the digits are also two valued. This suggests the use of the binary system.

5.2 A consistent use of the binary system is also likely to simplify the operations of multiplication and division considerably. Specifically it does away with the decimal[2] multiplication table. ...

<div align="center">∞⊗∞⊗∞⊗∞⊗∞○</div>
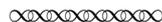
Predating the EDVAC, in 1938 Claude Shannon (1916–2001) published a ground-breaking paper "A Symbolic Analysis of Relay and

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[2]base ten

Switching Circuits" [29] in which he demonstrated how electronic circuits can be used for binary arithmetic, and more generally for computations in Boolean algebra and logic. These relay contacts and switches performed at speeds slower than vacuum tubes. A second project "Arithmetic Backwards from von Neumann to the Chinese Abacus" [24] examines the work of Shannon as well as representing numbers in base sixteen. While binary arithmetic is more efficient than base ten operations, communicating numbers as long strings of zeros and ones can be tedious for humans. Base sixteen allows an alternate to this in a base that is easily converted to base two, since sixteen is a power of two. The project continues with the observation that a Chinese abacus, when used to its fullest potential, can be used to perform calculations in base sixteen. This does not follow the historical record, but is an enrichment exercise. For more details, consult the project itself.

3.3 Sums of Numerical Powers: Archimedes Sums Squares in the Sand

Both discrete mathematics and calculus courses typically engage the topic of sums of numerical powers $\sum_{i=1}^{n} i^k$. For calculus these sums are key to integration of the power function x^k, especially before the fundamental theorem of calculus in the later seventeenth century. They range from Archimedes' third century B.C.E. determination of the area of a parabolic segment, through classical Greek and Indian mathematics on sums of cubes, to Arabic determination of sums of fourth powers for finding the volume of paraboloids of revolution, and then to Pierre de Fermat (1601–1665), Blaise Pascal (1623–1662), and others determining areas under "higher parabolas". But in a calculus course today, closed formulas for these sums are typically pulled out of hats like rabbits, and not justified in a manner meaningful to students.

In a discrete mathematics course, studying sums of powers creates one of the first triumphs of the subject, the realization that arbitrarily long sums obey patterned closed polynomial formulas, and that these can be proven by mathematical induction once they have been "guessed". Indeed these are often the initial exercises for students developing facility with induction as a proof technique. The patterns connecting these formulas for different values of the power k form what Fermat called "perhaps the most beautiful problem of all arithmetic", leading through the work of Jakob Bernoulli (1654–1705) [28] to the Euler-MacLaurin summation formula for sums of infinite series, Euler's (1707–1783) determination thereby that the sum of the reciprocal squares $\sum_{i=1}^{\infty} \frac{1}{i^2}$ is exactly $\pi^2/6$, and ultimately to modern research on the distribution of prime numbers [17].

The project "Sums of Numerical Powers in Discrete Mathematics:

Archimedes Sums Squares in the Sand" [27] studies just the first step in this long and fascinating story, and has been used with students in both lower division discrete mathematics and calculus. The project is quite flexible, its only prerequisite is basic algebra, and the instructor can select from various activities offered for a one to two week project.

In his treatise *On Conoids and Spheroids*, and as Proposition 10 in his treatise *On Spirals* [1, v. 11, p. 456][12, p. 122], Archimedes states, proves, and applies his claim about a sum of squares:

<div style="text-align:center">∞∞∞∞∞∞∞∞</div>

> If a series of any number of lines be given, which exceed one another by an equal amount, and the difference be equal to the least, and if other lines be given equal in number to these, and in quantity to the greatest, the squares on the lines equal to the greatest, plus the square on the greatest and the rectangle contained by the least and the sum of all those exceeding one another by an equal amount will be the triplicate of all the squares on the lines exceeding one another by an equal amount. [12, p. 122]

<div style="text-align:center">∞∞∞∞∞∞∞∞</div>

This rich one-sentence excerpt forms the entire primary source for the project. Students are led via a sequence of guided discovery exercises, often open-ended, to decipher the language of Archimedes, see why his claim is true, and to place it in the larger context either of integration or of finding patterns in formulas for sums of higher powers.

By 'line' Archimedes means a line segment. By the 'square on a line' he means a square whose side length is the length of the line, and by the 'rectangle contained by two lines' he means a rectangle with its two perpendicular sides given by their lengths. One challenge the project provides is interpreting Archimedes' verbal descriptions, and use of manipulative rods or graph paper has been encouraged productively by some instructors. Archimedes' highly verbal statement and proof actually has a beautiful geometric interpretation, discovered by a student and incorporated into the project as Figure 1.

After a series of warm-up exercises on sums of natural numbers (first powers), students engage Archimedes via exercises such as the following:

<div style="text-align:center">∞∞∞∞∞∞∞∞</div>

Exercise 1. As before, draw a picture of the two geometric constructs that Archimedes is saying 'will be' each other. In your picture, go ahead and make the simplifying choice that the smallest line represents the unit of length. Archimedes' claim will be that two constructs made of rectangles have the same total areas.
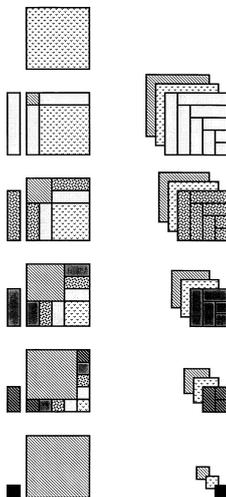
Figure 1. Pictorial representation of Archimedes' analysis of a sum of squares

Exercise 2. Archimedes provides a quite elaborate proof of his claim, with no picture. ... Figure 1 may be what was in his mind as he was drawing in the sand more than two thousand years ago. Mark up the displayed picture to explain why it proves what Archimedes claims. Notice that it only proves the claim for a particular number of 'lines'. How many? The picture nonetheless should convince you that the claim is true for any number of lines. Explain why it does that. Could you draw the picture necessary to prove it for nine lines? nineteen thousand lines? Explain why you are sure of that. The fact that you are convinced means that this has the nature of a 'proof by generalizable example', which was a common method of proof in mathematics until perhaps one or two hundred years ago. However, as mathematics became more highly developed, elaborate proofs by this method became less acceptable, because they rely on an intuitive sense that the example generalizes, but intuition can lead one astray or not be the same to everyone when things are complicated. Thus today we require proofs that do not rely just on the reader's acceptance of the intuitive generalization of an example.

Exercise 3. From Archimedes' claim deduce an equality involving $\sum_{i=1}^{n} i^2$, and therefrom deduce a closed formula for a sum of squares. Feel free to use the formula you already proved earlier

for $\sum_{i=1}^{n} i$. Is your new closed formula a polynomial? What degree is it? What is its leading coefficient? What is its constant term?

Exercise 4. Prove your formula for a sum of squares by generalizing algebraically the telescoping sum approach used earlier for a sum of natural numbers. Notice that you will again use your knowledge of the formula for a sum of natural numbers. For the sum of natural numbers you interpreted this telescoping approach geometrically using nested squares and gnomons. Discuss what would be involved geometrically to do this for your telescoping proof for a sum of squares.

Exercise 5. Recall that we saw that with a closed formula for arbitrary sums of squares in hand, we (and Archimedes) might be able to calculate the area of a region bounded on one side by a parabola. In particular, we determined that the area of the region that we studied underneath the parabola will be the number approached by $\frac{\sum_{i=1}^{n} i^2}{n^3}$ as n grows larger and larger. Now that you know a formula for a sum of squares, you can calculate this. Figure out exactly what that number being approached actually is, on the nose. Congratulations, you have computed the precise area of this curve-sided region. For those of you who have learned calculus, you know that this can also be obtained by antidifferentiation and the fundamental theorem of calculus; but to do so would be cheating by almost 2,000 years, wouldn't it, since you have just done it à la Archimedes, but the calculus approach was not discovered until the middle of the 17th century.

<div align="center">∞◎∞◎∞◎∞◎∞</div>
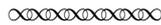
The project continues with exploration, conjectures, and some results on patterns for sums of higher powers. The goal of the project is for students to learn many basic notations, techniques, and skills in the context of an historically and mathematically authentic big motivating problem with multiple connections to other mathematics. Hopefully this will be much more effective and rewarding than simply being asked to learn various skills for no immediately apparent application. Many of the techniques first introduced in a discrete mathematics or calculus course arise naturally as needed in this project, like re-indexing summation notation, working with algebraic inequalities, and telescoping sums. Instead of separately covering such topics and techniques, that class time can simply be spent on the project, and students will learn those things in the process.

4 STUDENT RESPONSES, A COMMUNITY, AND FU-TURE DIRECTIONS

Over many years we have asked our students on post-course question-naires about the advantages and disadvantages of learning from primary sources. The disadvantages mentioned are surprisingly few and rare, and usually balanced with a concomitant advantage. For instance, some students say that the sources can be hard to read, but that it is worth it. Others have said that the sources don't provide a modern view, but that contrasting both the older and modern views is highly beneficial. On the other hand, the advantages given by our students are many and frequent, and include the following:

- "For me, being able to see how the thought processes were developed helps me understand how the actual application of those processes work[s]. Textbooks are like inventions without instruction manu-als."
- "The original sources can be debated to form new interpretations."
- "As a student you get to see where the math we do today came from and engage in the kind of thinking that was necessary to create it.
- "We learn directly from the source and attempt to learn concepts based off of the original proofs rather than interpretation of the original proof from someone else."
- "It gives you the sense of how math was formed which prepares you for how to think up new, innovative mathematics for the future."

The behavior of our students in the classroom, and their writing on project work, supports these comments. For example, after reading an historical source, students are more willing to ask questions about how to set up a problem and about the meaning of modern notation. Also, student written work contains more verbal justification than solutions prepared by students who have not studied from historical sources.

The interested reader will want to know that there is a vibrant world-wide community [15] devoted to incorporating history in the teaching of mathematics, including the approach espoused here of students engag-ing primary sources directly, and to making such sources accessible for students and teachers. The modules in our three case studies show that core content in standard courses for mathematics majors and other lower division courses can be covered by replacing standard textbook material with historical projects based on primary sources.

As we developed more and more modules, we realized that we could teach entire courses with them, jettisoning standard textbooks entirely. Indeed, in the past few years we have taught courses in discrete math-ematics, mathematical logic, and combinatorics based wholly on histor-

ical projects, needing no textbook. A paper describing how this was accomplished with discrete mathematics is currently in preparation; the dream that all mathematics might be taught and learned through direct student engagement with primary historical sources is discussed in [26].

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