CHAPTER 3

Analysis: Calculating Areas and Volumes

3.1 Introduction

In 216 B.C.E., the Sicilian city of Syracuse made the mistake of allying itself with Carthage during the second Punic war, and thus was attacked by Rome, portending what would ultimately happen to the entire classical Greek world. During a long siege, soldiers of the Roman general Marcellus were terrified by ingenious war machines defending the city, invented by the Syracusan Archimedes, greatest mathematician of the ancient world, born in 287 B.C.E. These included catapults to hurl great stones, as well as ropes, pulleys, and hooks to raise and smash Marcellus's ships, and perhaps even burning mirrors setting fire to their sails. Finally, though, probably through betrayal, Roman soldiers entered the city in 212 B.C.E., with orders from Marcellus to capture Archimedes alive. Plutarch relates that "as fate would have it, he was intent on working out some problem with a diagram and, having fixed his mind and his eyes alike on his investigation, he never noticed the incursion of the Romans nor the capture of the city. And when a soldier came up to him suddenly and bade him follow to Marcellus, he refused to do so until he had worked out his problem to a demonstration; whereat the soldier was so enraged that he drew his sword and slew him" [93, p. 97]. Despite the great success of Archimedes' military engineering inventions, Plutarch says that "He would not deign to leave behind him any commentary or writing on such subjects; but, repudiating as sordid and ignoble the whole trade of engineering, and every sort of art that lends itself to mere use and profit, he placed his whole affection and ambition in those purer speculations where there can be no reference to the vulgar needs of life" [93, p. 100]. Perhaps the best indication of what Archimedes truly loved most is his request that his tombstone include a cylinder circumscribing a

PHOTO 3.1. The death of Archimedes.

sphere, accompanied by the inscription of his amazing theorem that the sphere is exactly two-thirds of the circumscribing cylinder in both surface area and volume!

Contrast this with what one typically learns about areas and volumes in school. One is told that the area inside a circle is πr^2, with π itself remaining mysterious, that the volume of a sphere is $\frac{4}{3}\pi r^3$, and perhaps that its surface area is $4\pi r^2$. (The reader can easily convert this information into Archimedes' grave inscription.) Are these formulas, all discovered and proved by Archimedes in the aesthetically pleasing form stated at his grave, perhaps quite literally the only areas or volumes the reader "knows" for regions with curved sides? If so, is this because these particular geometric objects are incredibly special, while the precise areas and volumes of all other curved regions remain out of human reach, their curving sides making exact calculations unattainable?

The two-thousand-year quest for areas and volumes has yielded many strikingly beautiful results, and is also an extraordinary story of discovery. This challenge led to the development of integral calculus, a systematic method for finding areas and volumes discovered in the seventeenth century. (Excellent references for this development are [8, 21].) Calculus became the driving force of eighteenth-century mathematics, with far-reaching applications, especially involving motion and other types of change. Subsequent efforts to understand, justify, and expand its applicability spurred the modern mathematical subject known as "analysis" [20, 77, 83, 93, 97]. Our sources will trace the development of the calculus through the problem of finding areas. And we shall encounter philosophical controversies that boiled at the heart of mathematics for two millennia, with a dramatic turn of events as late as 1960.

Cultures prior to classical Greece amassed much knowledge of spatial relations, including some understanding of areas and volumes, motivated by endeavors such as resurveying and reshaping fields after floods, measuring the volume of piles of grain for taxation or community food planning, or determining the amounts of needed materials for constructing pyramids, temples, and palaces. Their knowledge resulted largely from empirical investigations, or inductive generalization by analogy, from simpler to more complicated situations, without rigorous deductive proof [91]. Babylonian and Egyptian mathematics was greatly restricted by mostly considering concrete cases with definite numbers, rather than general abstract objects [91]. It was the emerging mathematical culture of ancient Greece that embraced abstract concepts in geometry, and combined them with logic to implement deductive methods of proof with great power to discover and be certain of the truth of new results [84, 85]. Probably introduced by Thales around 600 B.C.E., these methods were firmly established by Pythagoras and his school during the sixth century.

The Pythagoreans' discovery that the diagonal of a square is incommensurable with its side greatly influenced the course of classical Greek mathematics. What they found is that no unit, however small, can be used to measure both these lengths (hence the term incommensurable; today we would say that their ratio is "irrational"). And yet to Greek mathematicians it was the very essence of a number to be expressed as some multiple of a chosen unit (i.e., a natural number). While they were able to use units to work with ratios comparing natural numbers (i.e., fractions), the discovery that this sort of comparison was impossible in general for geometric lengths caused them to reject numbers alone as sufficient for measuring the magnitudes of geometry. The study of areas was therefore not pursued by assigning a number to represent the size of each area, but rather by directly comparing areas, as on Archimedes' gravestone.

Perhaps the first solution to an area problem for a region with curved edges was Hippocrates of Chios's theorem in the mid–fifth century B.C.E., about a certain lune, the crescent-moon-shaped region enclosed by two intersecting circles (Figure 3.1) [45, pp. 17 f.]. Here semicircles and their diameters are arranged as shown, with CD perpendicular to the large diameter AB at its center. Hippocrates showed that the lune $AECFA$ equals ("in area," we would say) triangle ACD (Exercise 3.1). To accomplish this, he needed the theorem that the ratio (of areas) of two circles is the ratio of the squares of their diameters, which was part of a large body of deductive knowledge already familiar to him [93, Ch. 2].

FIGURE 3.1. A lune squared by Hippocrates

Hippocrates' result is called a "quadrature," or "squaring," of the lune, since the idea was to understand the magnitude of an area by equating it to that of a square of known size, i.e., one whose side was a "known" magnitude. A known quantity was one that one could construct using compass and straightedge alone, or equivalently, with the postulates of Euclid's *Elements* (see the geometry chapter). In practice, areas were often effectively squared by equating them to known triangles and rectangles, since with straightedge and compass one can always easily "square" the resulting triangle or rectangle [45, pp. 13 f.][84, 85].

Hippocrates also tried to "square the circle," i.e., construct a square with area precisely that of a given circle. Two other prominent construction problems from this era were "doubling the cube" (see the Introduction to the algebra chapter) and "trisecting the angle," i.e., constructing an angle one-third the size of a given angle. These are called the three classical problems of antiquity, and they motivated much of Greek mathematics for centuries, including investigations into what modifications of the construction rules might be necessary for solving them [98]. It is astonishing that all three took more than two thousand years to be solved, and the answers are all the same: each of the constructions is literally impossible with compass and straightedge alone (Exercises 3.2 and 3.3).

In the fifth century B.C.E. Democritus discovered that the volume of a cone is precisely one-third that of the encompassing cylinder with same base and height. We do not know how he derived these results, but we do know that he considered solids as possibly "made up" of an infinite number of infinitely thin layers, infinitely near together, for he said:

> If a cone were cut by a plane parallel to the base [and very close to the base], what must we think of the [areas of the] surfaces forming the sections? Are they equal or unequal? For, if they are unequal, they will make the cone irregular as having many indentations, like steps, and unevenness; but if they are equal, the sections will be equal, and the cone will appear to have the property of the cylinder, and to be made up of equal, not unequal, circles: which is very absurd [85, p. 119].

This paradox did not prevent exploitation of the idea that a solid is made up of these infinitely thin slices, or that a planar region such as a disk is made up of infinitely many infinitely thin parallel line segments. But the mathematical and philosophical foundations for such an approach would remain murky and controversial for two millennia. These infinitely thin objects would later be called "indivisibles" because, having no thickness, they cannot be divided. A certain schizophrenia ruled between the largely successful, but rigorously unsupported, use of indivisibles for calculating areas and volumes, and fully accepted rigorous deductive methods that were much more difficult to apply.

Related philosophical trends fueled the controversy. Plato distinguished between the objects of the mind and those of the physical world, e.g., an ideal mathematical circle versus a physical circle. In the context of the Democritean puzzle, this provoked questions about a distinction between indivisible physical and mathematical objects. Such questions were amplified when space and time were combined in the four paradoxes of Zeno of Elea (c. 450 B.C.E.). One of his paradoxes, the "arrow" [Aristotle, *Physics* VI, 9, $239^{\mathrm{b}}6$], refutes the existence of indivisibles. It claims that if an instant is indivisible, an arrow in flight cannot move during the instant, since otherwise the instant could be divided. But since time is made up only of such instants, the arrow can therefore never move.

A satisfactory mathematical response to this paradox did not gel until the nineteenth century. Classical Greek work sidestepped the dilemma by an incredibly ingenious approach avoiding indivisibles. Today called the "method of exhaustion," it was invented by Eudoxus in the fourth century B.C.E. to prove Democritus's discoveries on volumes of cones and pyramids. He proved results for curved figures by providing geometric approximations (usually with flat sides) whose area or volume was already known. If such approximations were found that together "exhaust" (i.e., fill out) the curved figure, then the exact result could often be verified. The method of exhaustion was featured around 300 B.C.E. in Book XII of the *Elements*, Euclid's compilation of much of Greek mathematics. Here Euclid proved the earlier results of Hippocrates and Democritus, as well as others, for instance that the ratio of the volumes of two spheres is the ratio of the cubes of their diameters (check this using the modern formula for the volume of a sphere). Euclid taught and wrote at the Museum and Library (something like a university) founded around that time in Alexandria by Ptolemy I Soter. It was the intellectual focal point of Greek scholarship for centuries, and eventually contained over 500,000 texts.

The exhaustion method was elevated to an art form by the great mathematician Archimedes of the third century B.C.E. He proved a comprehensive array of area and volume results, without equal until at least the seventeenth century. Our first original sources will come from two treatises of Archimedes on the area of a segment of a parabola. They reveal that while he felt obliged to provide proofs by the rigorous method of exhaustion, exemplified in *Quadrature of the Parabola*, his actual discovery tools, revealed in *The Method*, were full of the fruitful indivisibles excluded from formal proofs.

After Archimedes, several factors conspired to stymie further calculation of areas and volumes. One was the inability of Greek mathematics and philosophy to resolve the paradoxes attached to indivisibles, and associated issues about infinity and the nature of space and time. Second, Archimedes' *Method* was lost for two thousand years, requiring the rediscovery of its ideas by later generations. Creativity was also hampered by the meager interest the Roman conquerors of Greece showed in mathematics, science, or philosophy; their focus was on practical arts and engineering projects. Finally, it was difficult to press further with the method of exhaustion simply because it was cumbersome and not well suited to discovering new results.

The mathematics of classical Greece declined over several centuries, and by the fourth century C.E., Greek mathematical activity had effectively ceased. This end is marked by the barbaric murder in Alexandria of Hypatia, the first female mathematician we know much about, by a Christian mob [93]. The emergence of algebra in the Arabic world, with influences from Greek and Indian traditions, set the stage for later developments. After the founding of Islam by Mohammed (570–632), his followers made many military conquests, including Alexandria by 641, where they burned what books still remained in the great Library there. Yet within another century, a great cultural awakening in Islam established the "House of Wisdom" in Baghdad, comparable to the Museum and Library in Alexandria, with avid translation of Greek texts into Arabic (see the Introduction to the algebra chapter). One of the most important influences there was the mathematician Al-Khwarizmi, who in the early ninth century began to develop algebra, a general set of procedures for solving certain problems, and also wrote on the Hindu decimal numeration system with place values and zero [20, pp. 250 f.]. These advances made it possible to start generalizing arithmetic processes using algebraic symbols, and thus to calculate in more general situations [8, pp. 60 f.].

For many centuries Europe had little access to classical Greek mathematics, cut off by Arabic conquests. But in twelfth-century Spain and Sicily, Arabic, Greek, and Hebrew texts were translated into Latin and began to filter into Europe. Classical Greek mathematics, Arabic algebra, and the Hindu numeral system in Arabic texts began to affect European thinkers. In particular, the writings of the great classical Greek scholar Aristotle (384–322 B.C.E.) spurred lively argument among thirteenth and fourteenth century philosophers about the infinite, indivisibles, and continuity [8, 21].

An illustrative example of these discussions concerns two concentric circles. Each radial line cuts the circumference of each circle in exactly one point, setting up a one-to-one correspondence between the points on the larger circumference and those on the smaller. This suggests that the number of points on each circumference is exactly the same, and yet there would seem to be more on the outer one, since it is longer. The paradox that the longer curve has all its points in perfect correspondence with those of the shorter one was unresolvable to fourteenth-century thinkers, and would remain a puzzle for several more centuries, until mathematicians began to understand indivisibles. (See the set theory chapter for more on this.)

Medieval scholars also began a quantitative study of physical change, in particular the effect that continuous variation has on the ultimate form of something [8, pp. 81 f.][21]. The reader can see the connection to Democritus's ancient quotation on the volume of a cone: Each indivisible slice is a circular disk, but they vary continuously in area as we slide along the cone, ultimately combining to produce the volume of the cone. Exactly how quickly the disks vary in area will be crucial to measuring the total volume. Medieval study of how variation influences form is called the "latitude of forms"; while it was primarily philosophical, it opened the door to later mathematical work, and pioneered perpendicular coordinate axes as a way of representing change such as velocity and acceleration.

Aristotle's view, which heavily influenced early medieval thinkers, was dominated by a strong reliance upon sensory perception, and a reluctance to abstract and extrapolate beyond what was clearly derived from the physical world. He had argued that the process of successively dividing a magnitude could always continue further, and thus indivisibles could not exist. He also argued that because a number can always be increased, but the world is finite, the infinitely large is only potential, and not actual: "In point of fact they [mathematicians] do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish.... Hence, for the purposes of proof, it will make no difference to them to have such an infinite instead..." [20, p. 41].

However, a fifteenth-century trend toward Platonic and Pythagorean mysticism, accompanying the growth of humanism over the extremely rational and rigorous thought of earlier Scholastic philosophy, encouraged the previously denied use of the infinite and infinitesimal in geometry. As a result of this shift, mathematics was viewed as independent of the senses, not bound by empirical investigations, and thus free to use the infinite and infinitesimal, provided that no inconsistencies resulted. By the early seventeenth century this view opened up bold new approaches.

An early trailblazer was Johannes Kepler (1571–1630), a German astronomer and mathematician [8, pp. 108 f.][21, pp. 106 f.][77, pp. 11 f.]. He deduced his three famous laws of planetary motion from the incredibly detailed astronomical observations made by Danish astronomer Tycho Brahe prior to the availability of the telescope. Kepler's work supported Copernicus's heliocentric model for the solar system, demonstrating that the planets follow simple elliptical orbits around the sun. His laws describe these motions, including the relationship between the period of each orbit and the size of its ellipse. One of the greatest triumphs of the newly invented calculus later in the century was Isaac Newton's (1642–1727) mathematical derivation of Kepler's laws from Newton's own physical theories of force and gravitation.

Kepler made free use of indivisibles in both astronomical work and a treatise on measuring volumes of wine casks. He went far beyond the practical needs of the wine business, and wrote an extensive tract on indivisible methods. Two illustrative examples are his approaches to the areas of a circle and an ellipse (Exercise 3.4) [20, pp. 356 f.].

One of Kepler's contemporaries was the great Italian mathematician, astronomer, and physicist Galileo Galilei (1564–1642), the first to use a telescope to study the heavens, and often considered the founder of the experimental method and modern physics. A university professor at Pisa and Padua, Galileo deduced the laws of freely falling bodies and the parabolic paths of projectiles, initiating an era of applications of mathematics to physics. In his book *Two New Sciences*, he used indivisible methods to study the motion of a falling body, and he planned, but never wrote, an entire book on indivisibles [77, pp. 11–12],[93]. Galileo established mathematical rationalism against Aristotle's approach to studying the universe, and insisted that "The book of Nature is...written in mathematical characters" [49, vol. 19, p. 640]. His support of the Copernican theory that the earth orbits the sun caused his prosecution by the Roman Catholic Inquisition, forcible recantation of his ideas, and house arrest for the last eight years of his life. In 1979, Pope John Paul II appointed Vatican specialists to study the case, and after thirteen more years, in a 1992 address to the Pontifical Academy of Sciences, the Pope officially claimed the persecution of Galileo had resulted from a "tragic case of mutual incomprehension" [57]. Thus it took 350 years for the church to "rehabilitate" Galileo, and *Two New Sciences* remained on the church Index of forbidden books until 1822.

We will examine a text using indivisibles by Bonaventura Cavalieri (1598–1647), a pupil and associate of Galileo who also became a university professor. Cavalieri combined indivisibles with emerging algebraic techniques to produce many new insights into the problem of finding areas and volumes.[1] Archimedes had determined the area of a parabolic segment, which we will see is equivalent to finding the area bounded above by a portion of the curve $py = x^2$ (p a constant) and below by the x-axis. Cavalieri managed to generalize this to calculate areas bounded by "higher" parabolas $py = x^n$, although he had neither this modern notation nor our modern view of the curve as a functional relationship between two variables.

The explosion of techniques using indivisibles in the early to mid–seventeenth century is a good example of simultaneous independent discovery, leading to disputes about plagiarism and priority, exacerbated by mathematicians' reluctance to reveal their methods. Publishing was also difficult, since the first scientific periodicals came into existence only in the latter third of the century [77]. Much of the exchange of information occurred through correspondence with one man, Marin Mersenne, in Paris, who circulated the problems and manuscripts of others. Alternative approaches to higher parabolas and other curves and surfaces via indivisibles came from Cavalieri's fellow Italian Evangelista Torricelli (1608–1647),

[1] A method similar to Cavalieri's had already been used in China beginning in the third century to find the volume of a sphere, by comparing it to the "double box lid" obtained by intersecting two perpendicular cylinders inside a cube [116, pp. 282 ff.][165].

Frenchmen Pierre de Fermat (1601–1665), Blaise Pascal (1623–1662), Gilles Personne de Roberval (1602–1675), and Englishman John Wallis (1616–1703). Some of them were simultaneously pursuing the important "problem of tangents," which we will see united in the most spectacular way with the problem of finding areas.

Although certain curves defined by simple geometric relationships were inherited from the Greeks, the repertoire had been quite limited. Now there was a great expansion in the curves considered, such as the "cycloid" (the path traveled by a point on the edge of a rolling wheel), higher parabolas and hyperbolas, and the "catenary" formed by the shape of a hanging chain. There was an interplay with physics and optics, great interest in lengths of curves, and in properties of curves arising from mechanical motion, like that of a pendulum. Their analysis required a melding of geometry and algebra, overcoming the classical aversion to linking geometric measurements to numbers and to the algebraic equations that arose from studying numbers [77].

Francois Viète (1540–1603) had contributed much to the introduction of symbols into the previously largely verbal art of equations, making it easier to define and work with equations corresponding to geometrical constructions, and thus beginning to reconnect geometry with algebra. This direction expanded greatly in work of Fermat and René Descartes (1596–1650) (after whom Cartesian coordinates are named). Descartes worked on finding the line perpendicular to a curve at any point, and others such as Roberval and Fermat worked on the equivalent problem of finding the tangent to a curve at any point. To them this tangent line embodied the direction of motion of a point moving along the curve, or was interpreted as the line touching the curve at only that one point. Their ingenious methods were highly successful, but involved distances or magnitudes becoming vanishingly small, and thus controversy about their meaning and validity. Another problem studied with similar methods was that of "maxima and minima," i.e., finding the extreme values of a quantity or, in terms of curves, finding the highest or lowest point. Altogether the seventeenth century was an incredibly rich time of ferment and invention. (See also the Introduction to the number theory chapter.)

While there were indications in some of the new area and volume calculations that they were connected to the results of tangent problems, it was the next generation, specifically Isaac Newton, in England, and Gottfried Leibniz (1646–1716), in France and Germany, who transformed these indications into an explicit connection and exploited it as a tool for a tremendous leap. Working in the latter part of the seventeenth century, Newton and Leibniz not only explicitly recognized a connection between area and tangent problems that provided a general method for solving area, volume, and motion problems, but they also systematically explored the translation of such problems into formulaic guise, leading to a greater understanding of these phenomena in a more general and calculational setting. They invented a symbolic language to express and exploit these connections; thereby a new mathematical subject, the calculus, was born.

The beautiful relationship between areas and tangents will emerge as we read Leibniz's proof of the Fundamental Theorem of Calculus, providing his general way of solving area problems. The process involved in finding an area is today known as *integration* (or calculating an *integral*, as Leibniz himself called it). Using modern function terminology, if the graph of $y = f(x)$ is a curve lying above the x-axis between the vertical lines $x = a$ and $x = b$, then the area bounded by the curve, the axis, and the vertical lines is called the integral of f between a and b (draw a picture). The problem of finding tangents to curves is known as *differentiation* (finding a *derivative*), encoded in the relationship between infinitesimal changes along a curve, called *differentials,* another term due to Leibniz. These differentials, or *infinitesimals*, were a refinement of Cavalieri's indivisibles, and while their meaning and right to exist were vigorously attacked and debated, they were strikingly useful and successful, so concerns about their validity were subsumed by their efficacy. The essence of the Fundamental Theorem of Calculus of Leibniz and Newton is that integration and differentiation are inverse operations, and thus performing an integration is an antidifferentiation (i.e., inverse differentiation, or inverse tangent) problem.

There are two accompanying threads to follow as we explore whether the new calculus represented the complete solution to the area problem we set as our theme. First, how successful was it at solving problems, spawning new ones, and furthering mathematics and the physical and natural sciences? Here the answer is that the results were absolutely spectacular, quickly creating the "age of analysis" as the dominant force in mathematics and its applications through the entire eighteenth century. Second, what about the serious unresolved foundational questions? These difficult issues were in fact not resolved for more than another century, until the successes of the new analysis finally reached a point where the nagging foundational problems required resolution before further progress could be made.

Shortly after Leibniz's and Newton's discoveries, many other mathematicians embraced their methods, among them the distinguished Swiss Bernoulli brothers, Jakob (1654–1705) and Johann (1667–1748), and Leonhard Euler (1707–1783), Swiss mathematician and scientist extraordinaire, the most prolific author in all mathematical history. The Bernoullis translated geometrical and mechanical data into relationships between differentials, creating what we today call differential equations. Their results included lengths of curves of many new types, studying light caustics (curves occurring when light rays are reflected or refracted on curved surfaces, e.g., the beautiful curves formed on the surface of coffee with cream in a ceramic mug), and the form of sails blown by the wind. Two famous curve problems they solved were to find the shape of the catenary and the brachistochrone (the curve of shortest descent time for a bead moving down and across from one point to another, sliding without friction under the force of gravity, i.e., in some people's opinion the shape of the perfect ski slope). To their astonishment, the shape of the brachistochrone curve was the same as the solution to two other curve problems, the tautochrone and the cycloid. (See Simmons [157] for an exposition and wonderful exercises on these curves.)

Through the eighteenth century, calculus was enlarged into "mathematical analysis," with applications throughout the physical and natural sciences, primarily due to the vast and brilliant work of Euler. This involved a major shift away from working just with variable geometric quantities, towards working with functions to express specific relationships between variables. To Euler, "a function of a variable quantity is an analytic expression composed in whatever way of that variable and of numbers and constant quantities" [77]. In particular, he worked freely with infinite algebraic expressions, for instance beautiful and highly useful *infinite series* expressions like $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, which displays $\sin x$ as a polynomial of infinite degree, known today as a *power series* (see the Appendix (Section 3.8) for a brief introduction to infinite series). To Euler we owe our modern understanding of the power series representations of trigonometric, exponential, and logarithmic functions. His wizardry with infinite series makes inspiring reading [55]. Euler and others made incredible progress attacking an impressive variety of physical problems, for instance the physics and mathematics of elastic beams, vibrating strings, pendulum motion, projectile motion, water flow in pipes, and the motion of the moon, used for determining positions at sea.

During this exploration in what seemed like paradise, the creators of the new mathematical analysis often had to play fast and loose with the still rather shaky foundations of the calculus, but their intuition nonetheless guided them to wonderful results. To describe this vast expansion and its enormous impact on science could fill an entire book, but the reader may wish to look at [19, 81].

It slowly became clear that Euler's view of a function, as something given by an algebraic formula, was too narrow, especially for functions exhibiting periodic motion, such as for a vibrating string. An approach was developed that represented functions as infinite series built from various frequencies, by using the periodic trigonometric functions sine and cosine of multiples of the variable x, in place of the powers of x used for power series [75, 77]. These are named after one of their early nineteenth-century pioneers, Joseph Fourier (1768–1830), emerging particularly from his study of heat diffusion. Today Fourier series and their subsequent generalizations have become crucial tools in many diverse applications of mathematics to the physical world. But while the eighteenth-century explorations led to new types of functions and new ways to represent them, it also led directly back to the foundational questions left unresolved more than a century ear-

PHOTO 3.2. Huygens's cycloidal pendulum clock.

lier, with the persistent appearance of vanishingly small magnitudes or differences.

The missing understanding underlying various coalescing challenges was "What is a limit?" i.e., what did it mean for a quantity that depends on a variable to approach a limiting value as the variable itself approaches a certain number. No conclusive answer had been proposed yet, as we will discuss in the section on Leibniz's work.

This frustrating state of affairs was expressed by one of those who contributed to bringing rigor to analysis, the Norwegian Niels Henrik Abel (1802–1829), when he complained in an 1826 letter about

> the tremendous obscurity which one unquestionably finds in analysis. It lacks so completely all plan and system that it is peculiar that so many men could have studied it. The worst of it is, it has never been treated stringently. There are very few theorems in advanced analysis which have been demonstrated in a logically tenable manner. Everywhere one finds this miserable way of concluding from the special to the general and it is extremely peculiar that such a

procedure has led to so few of the so-called paradoxes [97, p. 947][1, vol. 2, pp. 263–65].

Despite a few eighteenth-century attempts, it was Augustin-Louis Cauchy (1789–1857) who finally substantially clarified the notion of limit and resolved many foundational difficulties, allowing analysis to develop further, by effectively obviating the necessity for slippery and evasive infinitesimals. Cauchy combined the notion of limit with those of variable and function to create a structure for calculus much like its present form in textbooks today. We now know that Cauchy's notion of what it means to approach a limiting value, and some of the important consequences of this understanding, were in essence independently formulated and published earlier by the Portuguese mathematician José Anastácio da Cunha (1744–1787) in 1782 and the Czech Bernard Bolzano (1781–1848) in 1817. However, their works were little noticed in the mathematical centers of France and Germany, and it is from Cauchy's work that modern analysis developed [93].

Cauchy presented his theory in lecture notes for his courses at the Ecole Polytechnique, in Paris, the new elite French institute of higher education and engineering. We will read selections leading to his formulation of the Fundamental Theorem of Calculus, illustrating shifts in point of view and more rigorous and unified understanding. For instance, we will see his thoroughly independent definitions of both the derivative and the integral of functions, unlike their circular intertwining in the work of Leibniz and Newton. Nonetheless, Cauchy did not break completely with the past, since he still attempted to interpret the infinitesimal within his new framework.

Cauchy's work synthesized the concepts and results of calculus with the deductive methods of ancient geometry, ushering in modern analysis, perhaps the largest and most fully developed branch of mathematics today.

Some of the most important unresolved questions of rigor still remaining in Cauchy's texts revolved around the notion of continuity. Intuitively, a function is continuous if its graph has no jumps; but does this mean, for instance, that all intermediate values are assumed? That is, if $f(x)$ is continuous for $a \leq x \leq b$, and if M lies between $f(a)$ and $f(b)$, is there necessarily some c between a and b for which $f(c) = M$? (Draw a picture!) This crucial property of continuity, upon which Cauchy's work relied, is called the Intermediate Value Theorem; its proof, provided independently by both Cauchy and Bolzano (see the set theory chapter), was still inadequate and demanded a deeper understanding of the real numbers themselves.

This emerged in the next fifty years, with the first actual definitions of the real numbers, founded only on the natural numbers, given by the German mathematicians Karl Weierstrass (1815–1897), Richard Dedekind (1831–1916), and Georg Cantor (1845–1918), who finally banished the seemingly useful fiction of the infinitesimal, in the so-called arithmetization of analysis. The proper foundation for the real numbers allowed a completely rigorous proof of the Intermediate Value Theorem, via the "completeness" property the real numbers possess, which ensures that they have no holes or gaps: specifically, any shrinking sequence of intervals must contain a point (real number) in common; i.e., one cannot shrink down to find an empty spot with no number there. One then also avoids the circularity inherent in statements like Cauchy's "an irrational number is the limit of the various fractions which provide values that approximate it more and more closely," in which his claim begs the very existence of irrational numbers.

This understanding of the real numbers progressed hand in hand with a broadening of the setting for analysis by mathematicians such as Bernhard Riemann (1826–1866) and Peter Lejeune-Dirichlet (1805–1859), for example, seeking detailed understanding of how to represent quite general functions by infinite series (see Appendix), and integrating functions with many, even infinitely many, points of discontinuity in an interval. Riemann slightly generalized Cauchy's definition of integration, in such a way that he could characterize precisely which functions could be integrated, while Dirichlet studied when a function could be represented by a Fourier series of trigonometric functions mentioned earlier. Thus began an intricate intertwining of the mutual development of the theories of integration, representations of functions by infinite series, and even of the very sense of what a function is. These efforts slowly grew beyond the original stimulus provided by the intuitive notion of area, but melded the emerging modern notions of continuity, discontinuity, variability, measurement of size of arbitrary sets of real numbers, integration, and functions and their representations by infinite series, ultimately becoming the modern synthesis we call real analysis. These nineteenth-century challenges in the development of real analysis directly motivated Georg Cantor's work on infinite sets featured in the set theory chapter.

By the turn of the twentieth century, researchers had discovered and tackled ever stranger functions. While some people wished simply to reject each new "pathology" discovered, others worked to incorporate them into the theory, and every time it emerged richer, more general, often more beautiful. The primary development, which set the stage for much of twentieth-century real analysis, was Henri Lebesgue's (1875–1941) introduction in 1902 of a totally new theory of integration, which we will explain and contrast with Cauchy's description of integration at the end of the section on Cauchy's work.

Meanwhile, other branches of analysis also emerged during the nineteenth century. Cauchy and his successors developed a beautiful theory of analysis for functions involving complex variables, i.e., those with both real and imaginary parts, with many applications today, e.g., to fluid flow and how an airplane wing provides lift. Other branches developed providing multivariable versions of all the important ideas; and today we even have a branch called functional analysis, in which functions themselves become mere points for other functions to be defined on, a metalevel analysis with applications in quantum physics.

At the close of the twentieth century, one of the hottest new fields in analysis is "wavelet theory," emerging from such applications as edge detection or texture analysis in computer vision, data compression in signal analysis or image processing, turbulence, layering of underground earth sediments, and computer-aided design. Wavelets are an extension of Fourier's idea of representing functions by superimposing waves given by sines or cosines. Since many oscillatory phenomena evolve in an unpredictable way over short intervals of time or space, the phenomenon is often better represented by superimposing waves of only short duration, christened wavelets. This tight interplay between current applications and a new field of mathematics is evolving so quickly that it is hard to see where it will lead even in the very near future [92].

We will conclude this chapter with an extraordinary modern twist to our long story. Recall that the infinitesimals of Leibniz, which had never been properly defined and were denigrated as fictional, had finally been banished from analysis by the successors of Cauchy in the nineteenth century, using a rigorous foundation for the real numbers. How surprising, then, that in 1960 deep methods of modern mathematical logic revived infinitesimals and gave them a new stature and role. In our final section we will read a few passages from the book *Non-Standard Analysis* [140] by Abraham Robinson (1918–1974), who discovered how to place infinitesimals on a firm foundation, and we will consider the possible consequences of his discovery for the future as well as for our evaluation of the past.

Exercise 3.1: Prove Hippocrates' theorem on the squaring of his lune.

Exercise 3.2: Research the history and eventual resolution of one of the three "classical problems" of antiquity.

Exercise 3.3: Find out what the "quadratrix of Hippias" is and how it was used in attempts to solve the problems of squaring the circle and trisecting the angle.

Exercise 3.4: Study Kepler's derivation [20, pp. 356 f.] of the area inside a circle. Critique his use of indivisibles. What are its strengths and weaknesses? Also study his matching of indivisibles to obtain the area inside an ellipse [20, pp. 356 f.]. Do you consider his argument valid? Why?

3.2 Archimedes' Quadrature of the Parabola

Archimedes (c. 287–212 B.C.E.) was the greatest mathematician of antiquity, and one of the top handful of all time. His achievements seem astounding even today. The son of an astronomer, he spent most of his life