as desired.

Exercise 5.8: Find a proof of Proposition 6 in Book II in the spirit of Euclid, which says: *If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.* Hint: See Figure 5.6.

Exercise 5.9: Proposition 11 in Book II states: *To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.* Which type of quadratic equation can be solved with the help of this proposition and how?

Exercise 5.10: Read about the golden ratio (or divine proportion, as it is sometimes called) and find its connection to Proposition 11.

Exercise 5.11: Figure 5.7 accompanies the proof of Proposition 11. Here, AB represents the given straight line. The cutting point H is constructed by Euclid as follows. First construct the square $ABDC$. Then bisect AC at E, and join E to B. Now extend CA until F, so as to make $EF = BE$. Construct the square $FGHA$ on AF, and extend GH to K.

Now complete the proof of Proposition 11 in the spirit of Euclid.

5.3 Cardano's Solution of the Cubic

Much of sixteenth-century Europe was in great turmoil. Wars and diseases such as the plague decimated the population and caused constant social and political upheavals. Nonetheless, it was a century of great artistic, literary, and scientific accomplishments. One of its renowned scholars was the Italian Girolamo Cardano. Known and sought after throughout Europe for his skills as a physician, he also distinguished himself as a natural philosopher, mathematician, and astrologer. Among his many books, one of the most wellknown today is his treatise on algebra, *Ars Magna* (The Great Art). However, his mathematical writings comprise only a small part of the ten volumes of his collected works, published in Lyons in 1663, under the title *Opera Omnia Hieronymi Cardani, Mediolanensis*. Cardano was a typical representative of the Renaissance. To quote from a biography:

FIGURE 5.6. Proposition 6.

FIGURE 5.7. Proposition 11.

PHOTO 5.6. *Ars Magna*, original 1545 edition.

> To Cardano, his scientific work is a means of understanding the world and human nature in general, as well as a way of gaining self-knowledge. He looks to science as an aid to orienting himself in a complex and dangerous world while being unable to effect any change. He wanted this analysis of the world and of himself to be scientific and rational; at the same time, he wanted it to satisfy the imagination as well as offer a way to comprehend the universe as a meaningful whole [60, pp. xxi ff.].

Cardano's father was a well-educated jurist and a friend of Leonardo da Vinci. He encouraged his son to study the classics and mathematics, as well as astrology. Defying the wish of his father he decided to become a physician rather than a lawyer. After receiving a doctorate in medicine from the University of Padua in 1526, he practiced medicine, first in a small town outside of Padua, later in Milan, where he simultaneously taught mathematics. By the middle of the century his fame as a physician had

PHOTO 5.7. Cardano.

spread throughout Europe, and he was summoned by nobility from as far away as Scotland for medical advice. In 1543 he obtained the chair of medicine at the University of Pavia, where he had begun his medical studies 23 years earlier. In 1562 he moved to the University of Bologna, where he remained until he was imprisoned by the Inquisition, among other things for casting the horoscope of Christ. Subsequently, he was banned from teaching, prohibited from publishing, and had to spend his final years in Rome. During his lifetime, Cardano published more than 200 works on a variety of topics, most of which went through many editions, including his autobiography *De Propria Vita Liber*, written during his final years in Rome. For biographical material on Cardano see [42, 60, 29].

To Cardano, mathematics was the language in which nature was to be described. This point of view becomes fully developed in the philosophy of Galileo two generations later, and is at the heart of the so-called scientific method, pioneered by Descartes in the following century. The *Ars Magna* was to be volume X in an encyclopedia of mathematics, which Cardano never completed and of which little remains [60, p. 73]. It is a treatise on algebraic equations, containing, besides some preparatory material, complete solutions for cubic equations as well as solutions to certain types of degree-four equations. In the introduction Cardano gives his version of the genesis of the solution for the cubic, after observing that the subject of algebra originates with al-Khwarizmi.

> In our own days Scipione del Ferro of Bologna has solved the case of the cube and first power equal to a constant, a very elegant and admirable accomplishment. Since this art surpasses all human subtlety and the perspicuity of mortal talent, and is a truly celestial gift and a very clear test of the capacity of men's minds, whoever applies himself to it will believe that there is nothing that he cannot understand. In emulation of him, my friend Niccolò Tartaglia of Brescia, wanting not to be outdone, solved the same case when he got into a contest with his [Scipione's] pupil, Antonio Maria Fior, and, moved by my many entreaties, gave it to me. For I had been deceived by the words of Luca Paccioli, who denied that any more general rule could be discovered than his own.[4] Notwithstanding the many things which I had already discovered, as is well known, I had despaired and had not attempted to look any further. Then, however, having received Tartaglia's solution and seeking for the proof of it, I came to understand that there were a great many other things that could also be had. Pursuing this thought and with increased confidence, I discovered these others, partly by myself and partly through Lodovico Ferrari, formerly my pupil. Hereinafter those things which have been discovered by others have their names attached to them; those to which no name is attached are mine [29, pp. 8–9].

Del Ferro (1465–1526) was a professor at the University of Bologna, and discovered an algebraic method for solving the equation $x^3+cx=d$, which he kept secret for reasons outlined in the introduction to the chapter until he disclosed it shortly before his death to Fior and his successor at Bologna, Annibale della Nave.

Reportedly, Tartaglia divulged his solution to Cardano in the form of a poem, after Cardano had sworn not to publish it in the book he was preparing at the time, since Tartaglia was planning on doing so himself.

> When the cube and its things near
> Add to a new number, discrete,
> Determine two new numbers different
> By that one; this feat
> Will be kept as a rule
> Their product always equal, the same,
> To the cube of a third
> Of the number of things named.
> Then, generally speaking,
> The remaining amount

[4]Cardano is probably referring to Paccioli's characterization of $x^4+bx^2=ax$ and $x^4+ax=bx^2$ as *impossible*.

PHOTO 5.8. Page from the *Ars Magna* on the equation $x^3 + cx = d$.

Of the cube roots subtracted
Will be your desired count [93, pp. 329–330].

The reader is invited to verify that in this way one indeed obtains a formula for the solution to the equation $x^3 + cx = d$ (Exercise 5.12). Upon publication of the *Ars Magna*, Tartaglia publicly accused Cardano of cheating him out of the fruits of his labor, and a lengthy argument ensued. Cardano, through his student Ludovico Ferrari, denied ever having taken such an oath. A detailed account can be found in the Foreword to [29]. As with most disputes of this nature it is difficult to sort out the truth. The following observations might be useful.

> In order to form any opinion on this matter one should take into account that Tartaglia was a man of obscure origin; not even his family name is known. "Tartaglia" means "the stutterer" and is a nickname. When the French pillaged Brescia in 1512, his mother sought refuge for her son in the church. But the soldiers also invaded the sanctuary, and the twelve-year-old boy was severely wounded by a sword cut: his jawbone was split, causing permanent damage. With enormous energy—and for the most part autodidactically—he worked to become a respected mathematics teacher and mechanical craftsman. He lived by his skills and regarded his knowledge as his personal property. He was not a scholar in the true sense, but he had great practical knowledge—in applied mathematics as well as in mechanics—and he was a very successful teacher. Cardano, on the other hand, was a highly esteemed physician as well as a universal scholar, although he had not yet come to fame. The two men were of totally different mentalities [60, pp. 8–9].

Here we examine in detail Chapter XII of the *Ars Magna* (The Great Art), the case "On the Cube Equal to the First Power and Number," or $x^3 = cx + d$. (Note that this case differs from the case $x^3 + cx = d$, discussed above).

Girolamo Cardano, from *The Great Art*

On the Cube Equal to the First Power and Number

Demonstration

Let the cube be equal to the first power and constant and let DC and DF be two cubes the product of the sides of which, AB and BC, is equal to one-third the coefficient of x, and let the sum of these cubes be equal to the constant. I say that AC is the value of x. Now since $AB \times BC$ equals one-third the coefficient of x, $3(AB \times BC)$ will equal the coefficient of x, and the product of AC and $3(AB \times BC)$ is the whole first power, AC having been assumed to be x. But $AC \times 3(AB \times BC)$ makes six bodies, three of which are $AB \times BC$ and the other three, $BC \times AB^2$. Therefore these six bodies are equal to the whole first power, and these [six bodies] plus the cubes DC and DF constitute

FIGURE 5.8. Cardano's demonstration.

FIGURE 5.9. Cardano's cube.

the cube AE, according to the first proposition of Chapter VI. The cubes DC and DF are also equal to the given number. Therefore the cube AE is equal to the given first power and number, which was to be proved.

It remains to be shown that $3AC(AB \times BC)$ is equal to the six bodies. This is clear enough if I prove that $AB(BC \times AC)$ equals the two bodies $AB \times BC^2$ and $BC \times AB^2$, for the product of AC and $(AB \times BC)$ is equal to the product of AB and the surface BE — since all sides are equal to all sides — but this [i.e., $AB \times BE$] is equal to the product of AB and $(CD + DE)$; the product $AB \times DE$ is equal to the product $CB \times AB^2$, since all sides are equal to all sides; and therefore $AC(AB \times BC)$ is equal to $AB \times BC^2$ plus $BC \times AB^2$, as was proposed.

Rule

The rule, therefore, is: When the cube of one-third the coefficient of x is not greater than the square of one-half the constant of the equation, subtract the former from the latter and add the square root of the remainder to one-half the constant of the equation and, again, subtract it from the same half, and you will have, as was said, a *binomium* and its *apotome*, the sum of the cube roots of which constitutes the value of x.

For example,

$$x^3 = 6x + 40.$$

Raise 2, one-third the coefficient of x, to the cube, which makes 8; subtract this from 400, the square of 20, one-half the constant, making 392; the square root of this added to 20 makes $20+\sqrt{392}$, and subtracted from 20 makes $20-\sqrt{392}$; and the sum of the cube roots of these, $\sqrt[3]{20+\sqrt{392}} + \sqrt[3]{20-\sqrt{392}}$, is the value of x.

Again,

$$x^3 = 6x + 6.$$

Cube one-third the coefficient of x, which is 2, making 8; subtract this from 9, the square of one-half of 6, the constant of the equation, leaving 1; the square root of this is 1; this added to and subtracted from 3, one-half the constant, makes the parts 4 and 2, the sum of the cube roots of which gives us $\sqrt[3]{4} + \sqrt[3]{2}$ for the value of x.

To the modern reader, Cardano's "demonstration" of the subsequent "rule" seems cumbersome, because of the use of geometry. There are many references to Euclid's *Elements* throughout the *Ars Magna*. Later mathematicians, in particular Viète and Descartes, developed a more efficient system of algebraic notation, completely freed from geometry (see, for instance, [22, pp. 84 ff.]).

What Cardano is proving in his demonstration of the rule for the equation $x^3 = cx + d$ is that it is sufficient to find quantities u and v such that $u^3 + v^3 = d$ and $3uv = c$. Then the solution is $x = u + v$. His proof becomes quite transparent if one uses a cube with side of length x, subdivided into eight "bodies," as indicated in Figure 5.9. From such a subdivision one can immediately deduce the Binomial Theorem for exponent 3, namely

$$(u + v)^3 = u^3 + 3uv^2 + 3u^2v + v^3.$$

This is none other than the "first proposition in Chapter VI" which he refers to. The main part of his argument is intended to show that

$$3uv^2 + 3u^2v = cx,$$

which is straightforward to reconstruct from the subdivided cube. Once he can show this, the desired formula follows easily, since he now is reduced to solving the system of equations

$$\begin{aligned} u^3 + v^3 &= d, \\ 3uv &= c. \end{aligned}$$

Its solution readily leads to the desired formula for x:

$$x = \sqrt[3]{d/2 + \sqrt{(d/2)^2 - (c/3)^3}} + \sqrt[3]{d/2 - \sqrt{(d/2)^2 - (c/3)^3}}.$$

(Why does this formula seem different from the one at the beginning of the Introduction?) Of course, if the expression under the square root in the formula is negative, that is, if $(d/2)^2 - (c/3)^3 < 0$, then one faces the awkward problem of not being able to evaluate the formula. This might not be so bad if it were to happen only with equations that have no real roots. But evaluation of Cardano's formula for the example

$$x^3 = 15x + 4,$$

which has the solution $x = 4$, leads to the expression

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

Basically, Cardano was at a loss how to deal with this case and dismisses it as absurd and useless in Chapter 37, where he deals with negative square roots [29, p. 220], [169, pp. 29–30]. Fortunately, this attitude did not prevail for long. Complex numbers were here to stay and already Rafaele Bombelli (1526–1572), in his influential algebra treatise published in 1572, taught how to calculate with cube roots of complex numbers. Del Ferro's and Cardano's formula thus gave great impetus to the development of a mathematically sound treatment of complex numbers. In the next section we will encounter one of the milestones along that road.

Exercise 5.12: Turn Tartaglia's poem into the formula in the Introduction.

Exercise 5.13: In Chapter XI of the *Ars Magna*, Cardano treats the case of the equation $x^3 + cx = d$. As he says at the beginning of the chapter:

> Scipio Ferro of Bologna well-nigh thirty years ago discovered this rule and handed it on to Antonio Maria Fior of Venice, whose contest with Niccolò Tartaglia of Brescia gave Niccolò occasion to discover it. He [Tartaglia] gave it to me in response to my entreaties, though withholding the demonstration. Armed with this assistance, I sought out its demonstration in [various] forms. This was very difficult [29, p. 96].

The rule Cardano refers to reads as follows:

> Cube one-third the coefficient of x; add to it the square of one-half the constant of the equation; and take the square root of the whole. You will duplicate this, and to one of the two you add one-half the number you have already squared and from the other you subtract one-half the same. You will then have a *binomium* and its *apotome*. Then, subtracting the cube root of the *apotome* from the cube root of the *binomium*, the remainder or that which is left is the value of x [29, pp. 98–99].

Find a demonstration for this rule in the spirit of Cardano. Hint: Use the picture in Figure 5.10.

5.4 Lagrange's Theory of Equations

The second half of the eighteenth century was not very favorably disposed toward pure mathematics. Since the time of Newton and Leibniz, the geometers, as mathematicians called themselves, were busily working on the development of the calculus. Euler's genius and phenomenal output defined the central problems and lines of development. The astonishing practical applications of the new theory left little time to catch one's breath and worry about the somewhat shaky foundations on which people juggled derivatives, integrals, and infinite series. There was much political and economic pressure to solve problems such as accurate navigation at sea, or construction of efficient turbines. Thus, there was neither livelihood nor prestige to be found in working on problems such as the theory of algebraic equations.

FIGURE 5.10. Cardano's rule.