Great Problems of Mathematics: A Summer Workshop for High School Students

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[College Mathematics Journal **25** (1994), 112-114]

Stimulating problems are at the heart of many great advances in mathematics. In fact, whole subjects owe their existence to a single problem that resisted solution. Nevertheless, throughout the curriculum we tend to present only polished theories and finished techniques, devoid of both the motivating problems and the long road to their solution. Why deprive our students of examples of the process by which mathematics is created? Why shield them from the central problems that have fired its development?

With these questions in mind, we designed a 3-week summer workshop for high school students, funded by the National Science Foundation under its Young Scholars program and held at Colorado College in 1992 and 1993. The 22 students who participated each summer were from across the country, were primarily beginning seniors, and had completed at least 2 years of algebra and 1 year of geometry.

For an introduction to sophisticated mathematics that minimizes the prerequisites, we are convinced that our approach is effective, gets students excited, and is as suited to college as to high school [1].

We examined the evolution of selected great problems from set theory, number theory, and calculus using original sources. Studying primary sources helps students to appreciate the progress over time in the clarity and sophistication of concepts and techniques. They also observe how progress can be blocked by certain ways of thinking, until some quantum leap takes us into a new era. Besides giving a firsthand look at the mathematical mindset of each period, no other method shows so clearly the evolution of mathematical rigor and the concept of what constitutes an acceptable proof.

We used primarily a traditional lecture approach in the first year, but in the second incorporated a combination of two pedagogical devices that proved amazingly effective: the "discovery" approach and daily writing.

The "discovery" method assumes that students should discover the mathematics for themselves. Hence, for each source we briefly provided the historical and mathematical context, alerted the students to any difficult points in the text, and then stood by to answer questions while they worked through the source in pairs. A wrap-up discussion let everyone share his or her understanding of the material, and any remaining difficulties were resolved. This method generated tremendous enthusiasm and a genuine sense of discovery. Strikingly, we saw that it also led to a deeper understanding of the sources than the lecture approach had achieved the previous year.

The students wrote frequently and about every aspect of the workshop: the mathematical details of the sources, their historical context, lecture notes from talks by guest speakers, thoughts jotted in the throes of problem solving, and their own ideas about the process that creates mathematics. Writing time was scheduled every day and, somewhat to our surprise, the students wrote prolifically and generally quite well. This writing experience led to a more comprehensive view of the great problems we studied as well as a much better grasp of mathematical details in their solutions.

To supplement the study of original sources, we discussed relevant problem solving techniques, and students paired up to choose research topics (e.g. Gödel's incompleteness theorem, surreal numbers, Diophantine equations) which they ultimately presented as a paper. Besides guest speakers, we included prose readings of works by Einstein, Plato and Poincaré.

After the workshop, the students almost unanimously reported a dramatic shift in their concept of mathematics, from seeing it as a mere tool for the sciences to respecting "pure" and continuing mathematical research. They now saw the inherent fascination of problems, not just the utility of the solutions.

The two versions of the workshop provided clear evidence of the inferiority of lectures to a careful combination of small group discussion and guided study. Moreover, the utility of daily writing (which of course is old news in the humanities) was vividly demonstrated as a tool for comprehending and mastering mathematics. Regardless of pedagogic technique, however, the ability of primary source material to engage students' attention and spur their efforts was dramatically evident.

A brief listing of our mathematical themes and original sources follows; [1] has a more detailed description with references.

Set Theory. Cantor's diagonal argument is the centerpiece of this topic. After his proof of the countability of the rational numbers (using a nonstandard order relation on the rationals) and the uncountability of the real numbers, we generalize the diagonal argument to produce infinite sets of ever larger cardinality. Finally, we discuss his statement of the Continuum Hypothesis and its independence from the axioms of set theory.

Fermat's Last Theorem. From the determination of all Pythagorean triples in Euclid's *Elements*, we leap to Euler's proof of Fermat's Last Theorem for exponent four, using Fermat's method of infinite descent. (Coincidentally, Andrew Wiles announced his proof of the theorem just before the second workshop ended.)

Area, volume, and the definite integral. Archimedes' computation of the area of a circle and Toricelli's computation of the volume of an infinite solid of revolution (using a precursor of the shell method) [2], leads to an excerpt from Cauchy's *Cours d'Analyse* defining the modern version of the definite integral, using the limit concept.

References

[1] R. Laubenbacher and D. Pengelley, Great problems of mathematics: A course based on original sources, *American Mathematical Monthly* **99** (1992) 313-317.

[2] Dirk J. Struik, *A Source Book in Mathematics, 1200-1800*, Princeton University Press, Princeton, 1986.