

The Bridge Between Continuous and Discrete

1.1 Introduction

In the early 1730s, Leonhard Euler (1707–1783) astonished his contemporaries by solving one of the most burning mathematical puzzles of his era: to find the exact sum of the infinite series $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots$, whose terms are the reciprocal squares of the natural numbers. This dramatic success began his rise to dominance over much of eighteenth-century mathematics. In the process of solving this then famous problem, Euler invented a formula that simultaneously completed another great quest: the two-thousand-year search for closed expressions for sums of numerical powers. We shall see how Euler's success with both these problems created a bridge connecting continuous and discrete summations.

Sums for geometric series, such as $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2$, had been known since antiquity. But mathematicians of the late seventeenth century were captivated by the computation of the sum of a series with a completely different type of pattern to its terms, one that was far from geometric. In the late 1660s and early 1670s, Isaac Newton (1642–1727) and James Gregory (1638–1675) each deduced the power series for the arctangent,¹ $\arctan t = t - \frac{t^3}{3} + \frac{t^5}{5} - \dots$, which produces, when evaluated at $t = 1$, the sum $\frac{\pi}{4}$ for the alternating series of reciprocal odd numbers $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$ [133, pp. 492–494], [135, pp. 436–439]. And in 1674, Gottfried Wilhelm Leibniz (1646–1716), one of the creators of the differential and integral calculus, used his new calculus of infinitesimal differentials and their summation (what we now call *integration*) to obtain the same value, $\frac{\pi}{4}$, for this sum by analyzing the quadrature, i.e., the area, of a quarter of a unit circle [133, pp. 524–527].

Leibniz and the Bernoulli brothers Jakob (1654–1705) and Johann (1667–1748), from Basel, were tantalized by this utterly unexpected connection

¹ This power series had also been discovered in southern India around two hundred years earlier, where it was likely derived for astronomical purposes, and written in verse [125], [133, pp. 494–496].

between the special number π from geometry and the sum of such a simple and seemingly unrelated series as the alternating reciprocal odd numbers. What could the connection be? They began considering similar series, and it is not surprising that they came to view the sum of the reciprocal squares, first mentioned in 1650 by Pietro Mengoli (1626–1686), as a challenge. Despite hard work on the problem, success eluded the Bernoullis for decades, and Jakob wrote, “If someone should succeed in finding what till now withstood our efforts and communicate it to us, we shall be much obliged to him” [258, p. 345]. The puzzle was so prominent that it became known as the “Basel problem.”

Around 1730, Euler, a student of Johann Bernoulli’s, took a completely fresh approach to the Basel problem by placing it in a broader context. He decided to explore the general discrete summation $\sum_{i=1}^n f(i)$ of the values of an arbitrary function $f(x)$ at a sequence of natural numbers, where n may be either finite or infinite. The Basel problem, to find $\sum_{i=1}^{\infty} \frac{1}{i^2}$, fits into this new context, since the sum can be written as $\sum_{i=1}^{\infty} g(i)$ for the function $g(x) = \frac{1}{x^2}$. Euler’s broader approach also encompassed an age-old question, that of finding formulas for sums of numerical powers, as we will now explain.

By the sixth century B.C.E., the Pythagoreans already knew how to find a sum of consecutive natural numbers, which we write as

$$1 + 2 + 3 + \cdots + n = \sum_{i=1}^n i^1 = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}.$$

Archimedes of Syracuse (c. 287–212 B.C.E.), the greatest mathematician of antiquity, also discovered how to calculate a sum of squares. Translated into contemporary symbolism, his work shows that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

Throughout the next two millennia, the search for general formulas for $\sum_{i=1}^n i^k$, a sum of consecutive k th powers for any fixed natural number k , became a recurring theme of study, primarily because such sums could be used to find areas and volumes. All these previous efforts also fit within Euler’s general context, since they are simply $\sum_{i=1}^n f_k(i)$ for the functions $f_k(x) = x^k$. While the function g for the Basel problem is very different from the functions f_k that produce sums of powers, Euler’s bold vision was to create a general approach to any sum of function values at consecutive natural numbers.

Euler’s aim was to use calculus to relate the discrete summation $\sum_{i=1}^n f(i)$ (with n possibly infinity) to a continuous phenomenon, the antiderivative, i.e., the integral $\int_0^n f(x)dx$. We know that these two provide first approximations to each other, since the sum can be interpreted as the total area of rectangles with tops forming a staircase along the curve $y = f(x)$, while the antiderivative, appropriately evaluated between limits, can be interpreted as the area

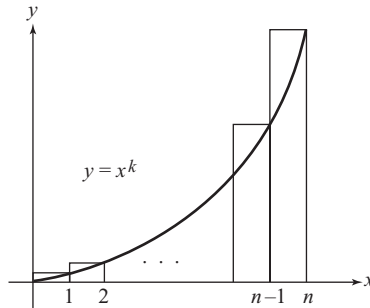


Fig. 1.1. Comparing a discrete sum with a continuous area.

under the curve itself (Figure 1.1). It is precisely the delicate difference, in both numerical value and in concept, between such discrete sums and continuous areas that mathematicians had in fact been exploring for so long when trying to find formulas for sums of powers.

For such powers the discrete sum is

$$\sum_{i=1}^n f_k(i) = \sum_{i=1}^n i^k = 1^k + 2^k + 3^k + \cdots + n^k,$$

while the continuous quantity for comparison is

$$\int_0^n f_k(x) dx = \int_0^n x^k dx = \frac{n^{k+1}}{k+1}.$$

Notice that the latter provides the first term in each of the polynomial summation formulas displayed above from the Pythagoreans and Archimedes. Understanding the dynamic between discrete and continuous amounts to quantifying exactly how separated, distinct, and finite objects blend with connected, homogeneous, and infinite spaces. Scholars as far back as Zeno, in classical Greece, grappled with this tension. Out of the fog of using discrete sums to approximate areas emerged the discovery of the differential and integral calculus in the seventeenth century. We shall see that Euler then turned the tables around in the eighteenth century by applying calculus to solve problems of the discrete.

Euler reconciled the difference between a discrete sum and a continuous integral via a striking formula using a corresponding antiderivative $\int_0^n f(x) dx$ as the first approximation to the summation $\sum_{i=1}^n f(i)$, with additional terms utilizing the iterated derivatives of f to make the necessary adjustments from continuous to discrete. Today we call this the *Euler–Maclaurin summation formula*. Euler applied it to obtain incredibly accurate approximations to the sum of the reciprocal squares, for solving the Basel problem, and these successes likely enabled him to guess that the infinite sum was exactly $\frac{\pi^2}{6}$. Armed

with this guess, it was not long before he found a proof, and announced a solution of the Basel problem to the mathematical world.

Euler's correspondents were greatly impressed. Johann Bernoulli wrote, "And so is satisfied the burning desire of my brother [Jakob], who, realizing that the investigation of the sum was more difficult than anyone would have thought, openly confessed that all his zeal had been mocked. If only my brother were alive now" [258, p. 345].

Euler also used his summation formula to provide closure to the long search for closed formulas for sums of powers. By now this thread had wound its way from classical Greek mathematics through the medieval Indian and Islamic worlds and into the Renaissance. Finally, during the Enlightenment, Jakob Bernoulli discovered that the problem revealed a special sequence of numbers, today called the *Bernoulli numbers*. These numbers became a key feature of Euler's summation formula and of modern mathematics, since, as we shall soon see, they capture the essence of converting between the continuous and the discrete. We will trace this thread through original sources from Archimedes to Euler, ending with Euler's exposition of how his general summation formula reveals formulas for sums of powers as well as a way to tackle the Basel problem. That Euler used his summation formula to resolve these two seemingly very different problems is a fine illustration of how generalization and abstraction can lead to the combined solution of seemingly independent problems.

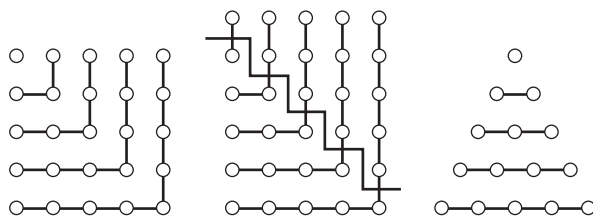


Fig. 1.2. Square, rectangular, and triangular numbers.

We return now to the very beginning of our story, which revolves around the relationship between areas and formulas for discrete sums of powers, such as the closed formulas above for the sums of the first n natural numbers and the first n squares. For the natural numbers it is not hard both to discover and to verify the formula oneself, but the Pythagoreans would not have written it as we do. For them, number was the substance of all things. Numbers were probably first represented by dots in the sand, or pebbles. From this, patterns in planar configurations of dots began to be recognized, and these were related to areas of planar regions, as in Figure 1.2 [18, p. 54f], [113], [133, p. 48ff], [135, p. 28ff], [258, p. 74ff].

In the figure, the arrangement and number of dots in each configuration suggests general closed formulas for various types of sums, illustrated by the three types $1+3+5+7+9 = 5^2$, $2+4+6+8+10 = 5 \cdot 6$, and $1+2+3+4+5 = (5 \cdot 6) / 2 = 15$. The reader may easily conjecture and prove general summation formulas with n terms for each of these.

For the third type, the total number of dots in the triangular pattern is clearly half of that in the rectangular pattern, which can be verified in general either algebraically, or geometrically from Figure 1.2. Thus we have deduced the closed Pythagorean formula above for the sum of natural numbers,² and we also see why the numbers $\frac{n(n+1)}{2}$ (i.e., 1, 3, 6, 10, 15, ...) deserve to be called *triangular numbers*. Notice that each of the three types of sums of dots has for its terms an *arithmetic progression*, i.e., a sequence of numbers with a fixed difference between each term and its successor. The first and third types always begin with the number one; the Pythagoreans realized that such sums produce *polygonal numbers*, i.e., those with dot patterns modeled on triangles, squares, pentagons, etc. (Exercises 1.1, 1.2).

The closed formula for a sum of squares, which we pulled from thin air earlier, is implicit in the work of Archimedes. At first sight it may seem unexpected that such a discrete sum should even have a closed formula. Once guessed, though, one can easily verify it by mathematical induction (Exercise 1.3). The formula arises in two of Archimedes' books [7]. In *Conoids and Spheroids* Archimedes develops and uses it as a tool for finding volumes of paraboloids, ellipsoids, and hyperboloids of revolution. In *Spirals* he applies it to obtain a remarkable result on the area enclosed by a spiral, stated thus in his preface:

If a straight line of which one extremity remains fixed be made to revolve at a uniform rate in a plane until it returns to the position from which it started, and if, at the same time as the straight line revolves, a point move at a uniform rate along the straight line, starting from the fixed extremity, the point will describe a spiral in the plane. I say then that the area bounded by the spiral and the straight line which has returned to the position from which it started is a third part of the circle described with the fixed point as the centre and with radius the length traversed by the point along the straight line during the one revolution.

² Another way of obtaining this formula occurs in a story about the developing genius Carl F. Gauss (1777–1855). When Gauss was nine, his mathematics teacher, J. G. Büttner, gave his class of 100 pupils the task of summing the first 100 integers. Gauss almost immediately wrote 5050 on his slate and placed it on his teacher's desk. Gauss had noticed that adding the numbers first in the corresponding pairs 1 and 100, 2 and 99, 3 and 98, ..., produced the sum 101 exactly 50 times, and then he simply multiplied 101 by 50 in his head. Fortunately, Büttner recognized Gauss's genius, and arranged for special tutoring for him. Gauss became the greatest mathematician of the nineteenth century [133, p. 654].

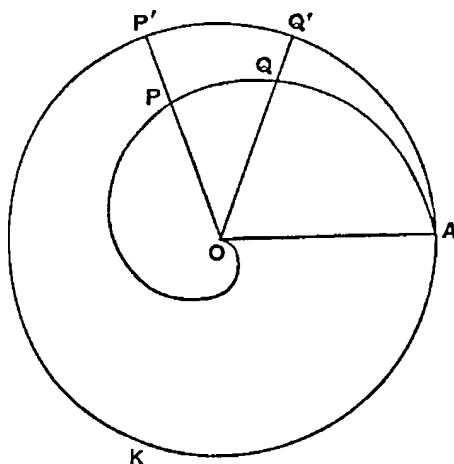


Fig. 1.3. Archimedes' area of a spiral.

Figure 1.3 illustrates Archimedes' claim that the area $OPQAO$ within the spiral is exactly one-third the area $AKP'Q'A$ of the "first circle."

Our original source will focus on Archimedes' expression for a sum of squares, and the resulting theorem on the area of the spiral, using the classical Greek *method of exhaustion*. Here we will see an early historical link between the discrete, in the form of the sum of squares formula, and the continuous, namely the area bounded by a continuous curve.

We will see also that Archimedes does not actually need an exact sum of squares formula to find the area in his spiral, but rather only the inequalities

$$\frac{n^3}{3} < \sum_{i=1}^n i^2 < \frac{(n+1)^3}{3},$$

which are highly suggestive of a more general pattern related to antidifferentiation of the k th-power functions $f_k(x) = x^k$ (Exercises 1.4, 1.5).

Our mathematical forebears were extremely interested in formulas for sums of higher powers $\sum_{i=1}^n i^k$, since they could use these to compute other areas and volumes. Let us pause to review from modern calculus how sums of powers are explicitly involved in the interpretation of the area under the curve $y = x^k$, for $0 \leq x \leq 1$, as the definite integral $\int_0^1 x^k dx$. Recall that to calculate this area from its modern definition as a limit of Riemann sums, we can subdivide the interval into n equal subintervals, each of width $1/n$, and consider the sum of areas of the rectangles built upwards to the curve from, say, the right endpoints of these subintervals, obtaining $\sum_{i=1}^n \frac{1}{n} \cdot \left(\frac{i}{n}\right)^k = \frac{1}{n^{k+1}} \sum_{i=1}^n i^k$ as an approximation to the area under the curve. The exact area is then the limit of this expression as n approaches infinity, since increasing n refines the accuracy of the approximation. Thus it is clear why having a closed formula for $\sum_{i=1}^n i^k$ (or perhaps just inequalities analogous to those of Archimedes

above) is key for carrying this calculation to completion. While this modern formulation streamlines the verbal and geometric versions of our ancestors, still the algebraic steps were essentially these.

As we continue to powers higher than $k = 2$, a formula for $\sum_{i=1}^n i^3$ jumps off the page once we compute a few values and compare them with our previous work. (The reader who wishes to guess the formula before we introduce it may consult Exercise 1.6 now.) It seems likely, from the work of the neo-Pythagorean Nicomachus of Gerasa in the first century C.E., that the mathematicians of ancient Greece knew this too; while it is not explicit in extant work, it is implicit in a fact about sums of odd numbers and cubic numbers found in Nicomachus's *Introductio Arithmetica* [19], [113, p. 68f] (Exercise 1.7).

The general formula for a sum of cubes first appears explicitly in the *Āryabhaṭīya*, from India [133, p. 212f], a book of stanzas perhaps intended as a short manual for memorization, which Āryabhaṭa wrote in 499 C.E., when he was 23 years old. Without any proof or justification, and in the completely verbal style of ancient algebra, he wrote:

The sixth part of the product of three quantities consisting of the number of terms, the number of terms plus one, and twice the number of terms plus one is the sum of the squares. The square of the sum of the (original) series is the sum of the cubes.

The earliest proof we have of the sum of cubes formula is by the Islamic mathematician Abū Bakr al-Karajī (c. 1000 C.E.), one of a group who began to develop algebra, in particular generalizing the arithmetic of numbers, centered around the House of Wisdom established in Baghdad in the ninth century [133, p. 251ff]. Al-Karajī's argument is noteworthy for its use of the method of "generalizable example" [113, p. 68f], [133, p. 255].

The idea of a generalizable example is to prove the claim for a particular number, but in a way that clearly shows that it works for any number. This was a common method of proof for centuries, in part because there was no notation adequate to handle the general case, and in particular no way of using indexing as we do today to deal with sums of arbitrarily many terms. Al-Karajī proves that $(1 + 2 + 3 + \cdots + 10)^2 = 1^3 + 2^3 + 3^3 + \cdots + 10^3$ in a way that clearly generalizes: He considered the square $ABCD$ with side $1 + 2 + \cdots + 10$ (Figure 1.4), subdivided into gnomons (L-shaped pieces) as shown, with the largest gnomon having ends $BB' = DD' = 10$. The area of the largest gnomon is 10^3 (the reader should carry this "calculation" out in a way that is convincing of "generalizability"). By the same generalized reasoning the area of the next-smaller gnomon is 9^3 , and so on for all the smaller gnomons, with only a square of side 1 left over.

Now one can think of the area of the large square in two ways. As the sum of gnomons it has area $1 + 2^3 + 3^3 + \cdots + 10^3$. On the other hand, as a square it has area $(1 + 2 + 3 + \cdots + 10)^2$. Today we would be inclined to use an algebraic proof by mathematical induction here; but it appears unnecessary if one sees how

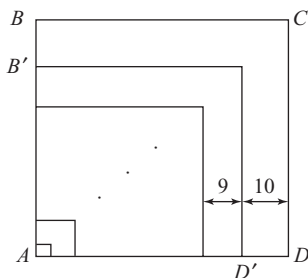


Fig. 1.4. Gnomons for the sum of cubes (not to scale).

to break the square up into gnomons, each identifiable numerically as a cube. This could all be done algebraically, although it would be excruciating, which is what leads us to use mathematical induction if we are invoking algebra rather than geometry (Exercise 1.8).

At this point we can be optimistic that for each fixed natural number k there is a polynomial in n for $1^k + 2^k + 3^k + \dots + n^k$. Based on our examples and the analogy to integration of x^k , the reader should try to guess the degree of the polynomial, the leading coefficient, and inequalities that might bound the polynomial like those of Archimedes. On the other hand, no general pattern is yet emerging for the details of the formula for various values of k , and worse, all the formulas we obtained emerged from ad hoc methods, each demanding separate verification.

The work of the Egyptian mathematician Abū ‘Alī al-Ḥasan ibn al-Haytham (965–1039) gives us the first steps along a path toward understanding these formulas in general [133, p. 255f]. He needed a sum of fourth powers in order to find the volume of a general paraboloid of revolution (in contemporary terms this involves integrating x^4). At that time, Islamic mathematicians were studying, rediscovering, and extending the work of Archimedes and others on volumes by the method of exhaustion. Ibn al-Haytham’s specific expression for fourth powers came from his equation (expressed here in modern symbolism) connecting sums of powers for different exponents:

$$(n + 1) \sum_{i=1}^n i^k = \sum_{i=1}^n i^{k+1} + \sum_{p=1}^n \left(\sum_{i=1}^p i^k \right).$$

Although ibn al-Haytham did not state a completely general result, rather only for $n = 4$ and $k = 1, 2, 3$, his proof, like al-Karajī’s, clearly generalizes for all n and k from his example, and uses a kind of mathematical induction (Exercise 1.9). In fact we can also prove his equation by interchanging the order of the double summation (Exercise 1.10). Letting $k = 3$, one can now obtain a formula for $\sum_{i=1}^n i^4$, as did ibn al-Haytham, by solving for it in his equation, first substituting the known formulas for smaller exponents. He did this, again by generalizable example. This is not quite as easy as we have made it sound, though, since in the process the double summation will actually give rise to the

very thing one is solving for again, in addition to its already stated occurrence. The reader may see how this actually works out in practice in Exercise 1.11.

Having followed in ibn al-Haytham's footsteps, we should now be reasonably convinced that in principle we could calculate a polynomial formula in terms of n for the sum $\sum_{i=1}^n i^k$ for any particular k . But we imagine that this will quickly become increasingly tedious and complicated with increasing k , and with no discernible pattern in the final formulas for different values of k . As our story unfolds, we will gradually uncover intricate patterns in these formulas reflecting the subtle connections between integration and discrete summation.

In the seventeenth century, the European creation of the calculus became a driving force in the development of formulas for sums of powers. In the second quarter of the century, a number of brilliant mathematicians had great success at *squaring* heretofore intractable regions (i.e., finding their areas), in particular the regions under the curves we write as $y = x^k$, which they called *higher parabolas*. Their successes, and especially the increasing use of *indivisible* methods, were the immediate precursors to the emergence of calculus later in the century. For instance, on September 22, 1636, Pierre de Fermat (1601–1665), of Toulouse, wrote to Gilles Persone de Roberval (1602–1675) that he could “square infinitely many figures composed of curved lines” [133, p. 481ff], including the higher parabolas. Roberval replied that he, too, could square all the higher parabolas using the inequalities

$$\frac{n^{k+1}}{k+1} < \sum_{i=1}^n i^k < \frac{(n+1)^{k+1}}{k+1}.$$

The reader is invited to confirm that these inequalities suffice for computing $\int_0^a x^k dx$ using our modern definitions (Exercise 1.12), and also, conversely, that Roberval's inequalities follow easily if we already know modern calculus (Exercise 1.13).

In reply to Roberval, Fermat claimed more, that he could solve “what is perhaps the most beautiful problem of all arithmetic” [19], namely finding the precise sum of powers in an arithmetic progression, no matter what the power. Fermat, apparently unaware of the works of al-Haytham, thought that the problem had been solved only up to $k = 3$, and stated that he had reached his results on sums of powers by using the following theorem on the *figurate numbers*³ derived from “natural progressions”:

The last number multiplied by the next larger number is double the collateral triangle;
 the last number multiplied by the triangle of the next larger is three times the collateral pyramid;

³ Fermat was likely also unaware of the work of Johann Faulhaber (1580–1635), who managed to develop explicit polynomials for $\sum_{i=1}^n i^k$ for all k up to 17. Faulhaber's interest and methods were also related to figurate numbers, but his work did not yield any general insight into the larger picture for all k [19].

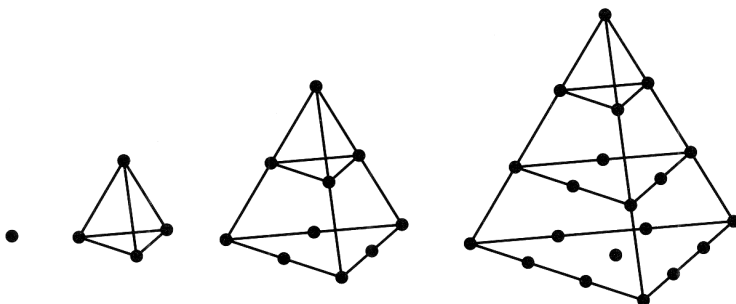


Fig. 1.5. Pyramidal numbers.

the last number multiplied by the pyramid of the next larger is four times the collateral triangulo-triangle; and so on indefinitely in this same manner [19].

By a “natural progression” Fermat simply means an arithmetic progression $1, 2, \dots, n$, whose “last number” is n . By the “collateral triangle” he means the triangular number (Figure 1.2) on a side with n dots. The figurate numbers then generalize this by counting dots in analogous higher-dimensional figures. For instance, by the “collateral pyramid” Fermat means to count the dots in a three-dimensional triangular pyramid on a side with n dots (Figure 1.5).

Fermat, typically, did not reveal his methods. But we can fill in the details of his claims by studying the figurate numbers and discovering their agreement with the numbers in the “arithmetical triangle”⁴ (Figure 1.6) of his contemporary and correspondent Blaise Pascal (1623–1662). This we will explore in our section on the work of Fermat and Pascal.

Fermat’s results on figurate numbers, and his derivation therefrom of formulas for sums of powers, could indeed be carried on indefinitely, but the process quickly becomes cumbersome and seemingly lacks insight. Despite Fermat’s enthusiasm for the problem, it appears at first that his procedure yields not much more than ibn al-Haytham’s. But what it did introduce was a major role for the figurate numbers that appear in the arithmetical triangle. And since the numbers in the arithmetical triangle have yet other important properties and patterns, namely in their roles as combination numbers and binomial coefficients, Fermat helped pave the way for future developments.

Blaise Pascal, in his *Treatise on the Arithmetical Triangle* [100, v. 30], made a systematic study of the numbers in his triangle, simultaneously encompassing their figurate, combinatorial, and binomial roles. Although these numbers had emerged in the mathematics of several cultures over many centuries [133], Pascal was the first to connect binomial coefficients with combinatorial coefficients in probability.

⁴ Today called Pascal’s triangle.

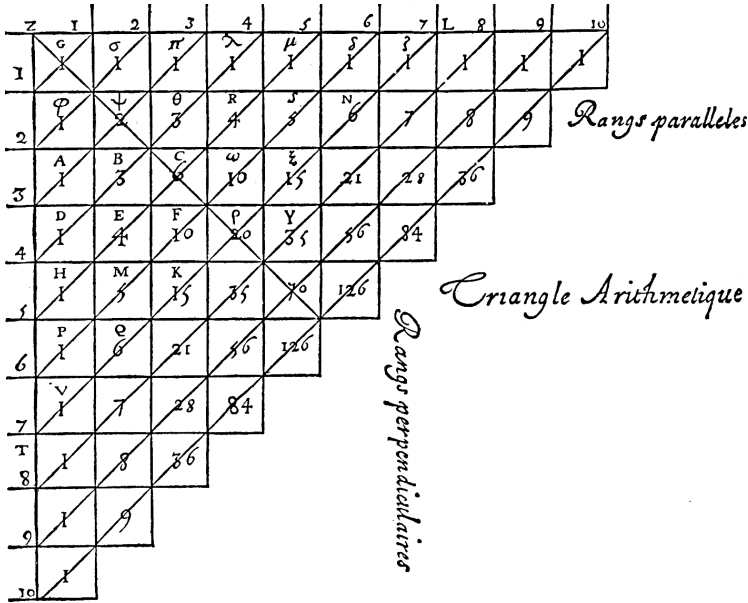


Fig. 1.6. Pascal's arithmetical triangle.

A major motivation for Pascal's treatise was a question from the beginnings of probability theory, about the equitable division of stakes in an interrupted game of chance. The question had been posed to Pascal around 1652 by Antoine Gombaud, the Chevalier de Méré, who wanted to improve his chances at gambling: Suppose two players are playing a fair game, to continue until one player wins a certain number of rounds, but the game is interrupted before either player reaches the winning number. How should the stakes be divided equitably, based on the number of rounds each player has won [133, p. 431, 451ff]? The solution requires the combinatorial properties inherent in the numbers in the arithmetical triangle, as Pascal demonstrated in his *Treatise*, since they count the number of ways various occurrences can combine to produce a given result.

Pascal also wrote another treatise, *Potestatum Numericarum Summa* (Sums of Numerical Powers), in which he presents his own approach to finding sums of powers formulas (he actually produces a prescription for much more general sums even than $\sum_{i=1}^n i^k$). We present his clearly written exposition. There we see that, armed with an ingenious idea based on the coefficients $\binom{m}{j}$ in the expansion of a binomial (i.e., $(a + b)^m = \sum_{j=0}^m \binom{m}{j} a^j b^{m-j}$), Pascal describes a procedure for finding sums of powers formulas. His final result is embodied in the equation

$$(k + 1) \sum_{i=1}^n i^k = (n + 1)^{k+1} - 1 - \sum_{j=0}^{k-1} \left[\binom{k+1}{j} \sum_{i=1}^n i^j \right].$$

Clearly one can solve here, if still tediously, for an explicit formula for the sum of k th powers, by using at each stage the already known formulas for lower exponents.

By this time in our story we will begin to discern some patterns in the sums of powers formulas for the first few values of k , which we can actually prove for general k from Pascal's equation. We can show that

$$\sum_{i=1}^n i^k = \frac{n^{k+1}}{k+1} + \frac{1}{2}n^k + \boxed{?}n^{k-1} + \cdots + \boxed{?}n + 0,$$

a $(k+1)$ st-degree polynomial in n with zero constant term, in which we know the first two coefficients (the second term actually has a nice geometric interpretation (Exercise 1.14), which suggests the sign of the third). This leads us to hope there is a pattern to the remaining coefficients, and to wonder what they might mean in the larger picture of the relationship between discrete summation, $\sum_{i=1}^n i^k = \frac{n^{k+1}}{k+1} + \cdots$, and continuous summation, $\int_0^x t^k dt = \frac{x^{k+1}}{k+1}$.

Jakob Bernoulli (1654–1705) discovered the general pattern in the polynomial formulas for sums of powers. We find him explaining it in a small section of his important treatise *Ars Conjectandi* (Art of Conjecturing) on the theory of probability. Since the combination numbers, figurate numbers, and binomial coefficients are the same, it is not surprising that Bernoulli's work on sums of powers occurs in his treatise on probability theory. He discerns a general pattern in the coefficients of the polynomials, writing them in terms of the combination numbers in the arithmetical triangle and a new sequence of special numbers, which he believes occur in a predictable way throughout all the formulas for summing powers.

These new numbers soon came to be called the *Bernoulli numbers*, and ever since, they have played an important role in mathematics. They are a sequence of rational numbers, which we will denote by $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, \dots , having a simple recursive law of formation. Bernoulli saw a pattern in the formulas in which these numbers seem to appear consistently. Specifically, he claimed, from calculating and examining the formulas explicitly up to the tenth powers, that the sums can be expressed as the following polynomials in n :

$$\begin{aligned} \sum_{i=1}^n i^k &= \frac{n^{k+1}}{k+1} + \frac{1}{2}n^k + \frac{k}{2}B_2n^{k-1} + \frac{k(k-1)(k-2)}{2 \cdot 3 \cdot 4}B_4n^{k-3} \\ &+ \frac{k(k-1)(k-2)(k-3)(k-4)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}B_6n^{k-5} \\ &+ \cdots + \text{ending in a term involving } n \text{ or } n^2. \end{aligned}$$

A critical observation here is that the Bernoulli numbers that occur are the same numbers in all the formulas, even as k varies. The pattern claimed here is clear (including that the odd Bernoulli numbers beginning with B_3 are

all zero, and that the constant term in each polynomial is always zero). Observe that by setting $n = 1$ on both sides of this family of equations, we obtain

$$1 = \frac{1}{k+1} + \frac{1}{2} + \sum_{j=2}^k \frac{1}{k+1} \binom{k+1}{j} B_j$$

for each $k \geq 2$. The k th equation clearly allows recursive calculation of B_k from knowing the previous Bernoulli numbers⁵ (Exercise 1.15).

Our chapter culminates by reading from the work of Leonhard Euler a few decades later. Euler dominated eighteenth-century mathematics, and produced seminal ideas in almost all its branches, as well as in physics. He was also perhaps the most prolific human writer of all time: his collected works are still in the process of being published, and will span close to one hundred thick volumes. Euler was particularly fascinated by the interplay between the continuous and the discrete in studying series, and the eighteenth century became a garden of discoveries about infinite series and related functions, largely thanks to Euler's genius [135, Chapter 20]. Euler's summation formula for series will bring together the sums of powers problem and the Basel problem on the infinite sum of reciprocal squares.

We have already mentioned Euler's early attraction to the famous Basel problem, to find the exact sum of the convergent series of reciprocal squares

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \boxed{?}.$$

In a series of papers through the 1730s and beyond, apparently initially motivated largely by desire to sum this series, Euler discovered, applied, and refined his summation formula for obtaining approximations to finite and infinite sums, paradoxically by using divergent series [66, v. 14]. Since his formula was also independently discovered by the Scottish mathematician Colin Maclaurin (1698–1746), it is today called the *Euler–Maclaurin summation formula*.

Around the year 1730, the 23-year-old Euler, along with his frequent correspondents Christian Goldbach (1690–1764) and Daniel Bernoulli (1700–1782) (son of Johann, Euler's teacher), tried to find more and more accurate fractional or decimal estimates for the sum of the series of reciprocal squares. They were likely trying to guess the exact value of the sum, hoping to recognize that their approximations looked like something familiar, perhaps involving π , as had Leibniz's series. But these estimates were challenging, since the series converges very slowly. To wit, if we estimate the sum simply by calculating a partial sum $\sum_{i=1}^n \frac{1}{i^2}$, we may be sorely disappointed by the accuracy achieved.

⁵ Explicit formulas for Bernoulli numbers, which do not rely on recursive knowledge about the previous numbers, are much more complicated [98].

The error is precisely the tail end of the series, which is bounded via

$$\frac{1}{n+1} = \int_{n+1}^{\infty} \frac{1}{x^2} dx < \sum_{i=n+1}^{\infty} \frac{1}{i^2} < \int_n^{\infty} \frac{1}{x^2} dx = \frac{1}{n},$$

by the standard method of inscribing and circumscribing rectangles of unit width along the curve $y = 1/x^2$. So if one were to add up 100 terms of the series by hand, as accurately as needed, the untallied tail end would be known only to lie between $1/100$ and $1/101$. Even taking this fully into account, the accuracy with which one would know the true sum would still only be the difference of these two numbers, i.e., about one ten-thousandth. Euler wanted far greater accuracy than this (Exercise 1.16). He first developed some clever special methods, and then in the early 1730s he hit gold with the discovery of his summation formula.

When we read Euler, we will see that his summation formula is in essence

$$\sum_{i=1}^n f(i) \approx C + \int^n f(x) dx + \frac{f(n)}{2} + B_2 \frac{f'(n)}{2!} + B_3 \frac{f''(n)}{3!} + B_4 \frac{f'''(n)}{4!} + \dots,$$

where $\int^n f(x) dx$ means a fixed antiderivative without the usual constant of integration added on, but with n substituted for x , and C denotes a constant that depends on f and the antiderivative chosen, but is independent of n . The motivation we can provide at this point is twofold. First, when f is specialized to the power functions f_k , Euler's formula clearly specializes to Bernoulli's sum of powers formulas (Exercise 1.17). Second, it is obvious that the first three terms in the formula correspond to the trapezoid approximation to the integral (Exercise 1.18). It is reasonable to expect that the difference between the discrete sum on the left and the area represented by the antiderivative on the right will involve how the graph of f curves, and hence the derivatives of f ; but the surprising thing is that these derivatives are all evaluated only at the single value n . We will see Euler derive his formula ingeniously from Taylor series.

One of Euler's first uses of his summation formula was to approximate the sum of the reciprocal squares. In a paper submitted to the St. Petersburg Academy of Sciences on the 13th of October, 1735, Euler applied it to approximate the sum of reciprocal squares and other series. He calculated the sum of reciprocal squares correct to twenty decimal places! Only seven and a half weeks later, Euler presented another paper, solving the famous Basel problem by demonstrating that the precise sum of the series is $\pi^2/6$. "Now, however, quite unexpectedly, I have found an elegant formula for $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}$, depending upon the quadrature of the circle [i.e., upon π]" [245, p. 261] (we paraphrase his proof in a footnote in the first of two sections on Euler's work). He even showed how to generalize his approach to find the exact sums of many other infinite series, such as the sum of the reciprocal fourth powers. While Euler's proof solving the Basel problem was soon criticized as lacking rigor,

he was understandably convinced of the truth of his answer, partly because it so perfectly matched the highly accurate approximation from his summation formula. Later he found other, rigorously acceptable, ways of justifying his claim.

We may never know with certainty whether Euler already suspected, when he wrote his paper of October 13, that the exact sum was $\frac{\pi^2}{6}$, or whether his calculation to twenty places was actually part of guessing the answer. We do know that Daniel Bernoulli wrote to him “The theorem on the sum of the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{pp}{6}$ and $1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{p^4}{90}$ is very remarkable. You must no doubt have come upon it a posteriori. I should very much like to see your solution” [10, p. 1075].

In our two sections on Euler’s work we will study the summation formula in his own words from his book *Institutiones Calculi Differentialis* (Foundations of Differential Calculus), published in 1755. Here his presentation of the formula is intertwined with many of his subsequent discoveries.

In reading Euler’s work, we will find that he ignores many questions we have about the rigor and validity of the mathematical steps he takes and the conclusions he draws. Not the least of these is that his summation formula usually diverges, yet still he calculates with great effectiveness using it. In this respect we should view Euler as a pioneer whose vision, brilliance, intuition, and experience about questions of convergence and divergence allowed him to excel where most mortals would stumble.

In our first selections from Euler’s book we will see him derive the summation formula, analyze the Bernoulli numbers it contains, and relate these numbers to familiar power series from calculus, proving many of the most intriguing properties of the Bernoulli numbers. Finally, he applies the summation formula to give the first actual proof for Bernoulli’s summation of powers formulas, thus completing the long search.

In our last section we will read how he uses his summation formula to make his remarkable approximation for the sum of reciprocal squares, before he proved that the value is $\pi^2/6$. Here as elsewhere Euler is always rechecking and verifying his results in different ways, with confirmation serving as his stabilizing rudder for confidence in further work.

In the *Institutiones* Euler also not only makes an exact determination of the sum of reciprocal squares as $\pi^2/6$, but actually finds the exact sums of all the series of reciprocal even powers, namely the series $\sum_{i=1}^{\infty} \frac{1}{i^{2k}}$ for every natural number k . Most unexpectedly, the very same Bernoulli numbers that help approximate these sums via Euler’s summation formula will occur one by one in the precise formulas for the sums of each of these series. This seems a striking coincidence, but actually hints at a link between Euler’s summation formula and Fourier analysis, a modern branch of mathematics that studies the representation of arbitrary functions as infinite sums of trigonometric functions of various frequencies [137, Chapter 14].

Thus wends the thread of the relationship between the continuous and the discrete through two millennia, from the ancient counting of a number of

dots in comparison to the area of a triangle through Euler's approximations of sums of series in relation to integration. We see the Bernoulli numbers emerge as key to this dynamic, and arise unexpectedly in other phenomena. Their importance in many parts of mathematics has grown continually ever since Euler. Today they permeate deep results in fields ranging from number theory to differential and algebraic topology [169, Appendix B], in addition to their ongoing importance in numerical analysis via the summation formula. We discuss this further at the end of the chapter. The link the Bernoulli numbers provide between the continuous and the discrete, first unveiled by Euler, continues to be key to advances in modern mathematics.

Exercise 1.1. In the spirit of the triangular and square numbers of Figure 1.2, generalize to define pentagonal numbers, hexagonal numbers, and general polygonal numbers for any regular polygon of side n . Deduce formulas showing that sums of terms in increasing integer arithmetic progressions beginning with 1 produce the polygonal numbers, and obtain closed formulas for these.

Exercise 1.2. Write out a table of polygonal numbers and discover some more patterns from this table. Prove your conjectures [258, p. 94].

Exercise 1.3. Verify the sum of squares formula

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

using mathematical induction. Perhaps discover or look up some other ways to obtain the formula that do not require knowing it in advance. Your proof using mathematical induction requires this advance knowledge, which is always a drawback of induction: the result needs to be known before proof by induction is possible.

Exercise 1.4. Verify that $\frac{n^3}{3} < \sum_{i=1}^n i^2 < \frac{(n+1)^3}{3}$ follows from the sum of squares formula. State and prove analogous inequalities for sums of zeroth and first powers. Then make a generalizing conjecture about analogous inequalities for $\sum_{i=1}^n i^k$ for any positive integer k . Verify your conjecture in various situations.

Exercise 1.5. Use polar coordinates to calculate the area inside Archimedes' spiral with the fundamental theorem of calculus, and compare it with his theorem.

Exercise 1.6. Guess a formula for sums of cubes: First calculate the first six sums. Then prove by mathematical induction that your guess is correct.

Exercise 1.7. Nicomachus wrote, "When the successive odd numbers are set forth indefinitely beginning with 1, observe this: The first one makes the potential cube; the next two, added together, the second; the next three, the third; the four next following, the fourth; the succeeding five, the fifth; the next six, the sixth; and so on" [180, Book 2, Chapter 20]. State and prove his

general pattern (Hint: average within the blocks), and then use it to obtain and prove the general formula for the sum of the first n cubes.

Exercise 1.8. Prove $(1 + 2 + \cdots + n)^2 = (1 + 2 + \cdots + (n - 1))^2 + n^3$ by mathematical induction, and discuss how the inductive step can be interpreted with the geometry of Al-Karajī's figure.

Exercise 1.9. Prove ibn al-Haytham's equation by mathematical induction. Perhaps first try his example values of n and k .

Exercise 1.10. Prove ibn al-Haytham's equation by interchanging the order in his double summation.

Exercise 1.11. Deduce the formula for a sum of fourth powers from ibn al-Haytham's equation, by inductively substituting the known formulas for smaller values of k .

Exercise 1.12. Calculate $\int_0^a x^k dx$ by considering lower and upper sums of rectangles based on left and right endpoints of equally spaced partitions of the interval, and by using Roberval's inequalities to compute the appropriate limit.

Exercise 1.13. Prove Roberval's inequalities by interpreting the sum of powers as both an upper and lower Riemann sum for an obvious function (you may use the calculus).

Exercise 1.14. By the time we have read Pascal's work we will be able to show (Exercise 1.38) that

$$\sum_{i=1}^n i^k = \frac{n^{k+1}}{k+1} + \frac{1}{2}n^k + \boxed{?}n^{k-1} + \cdots + \boxed{?}n + 0.$$

There is a simple geometric interpretation of the second term. Draw a picture illustrating the difference between the region under the curve $y = x^k$ for $0 \leq x \leq n$ and the region of circumscribing rectangles with ends at integer values. Interpreting their areas as $\int_0^n x^k dx = \frac{n^{k+1}}{k+1}$ and $\sum_{i=1}^n i^k$, find an interpretation in the picture of how the term $\frac{1}{2}n^k$ above represents part of the region between these two, and explain what its connection is to the trapezoid rule from calculus as a numerical approximation for definite integrals. This should suggest to you the sign of the next term in the formula above. What should it be and why?

Exercise 1.15. Use Bernoulli's recursive formulas to calculate the first several Bernoulli numbers. Use them to check Bernoulli's claim against the sums of powers for which you already know formulas. Also conjecture at least one further property it appears the Bernoulli numbers may have from what you find, and then calculate a few more numbers to begin testing your conjecture.

Exercise 1.16. Put yourself in Euler’s shoes and try making an educated guess for the exact sum of the reciprocal squares. First calculate a particular partial sum by hand to a certain accuracy (maybe to the tenth term for starters), bound the remainder with integrals, as in the text, and try averaging these to add to the partial sum to make a guess for the infinite sum. Then, with the sum $\frac{\pi}{4}$ of Leibniz’s series as inspiration, try dividing π by your guess, to see whether you obtain approximately a whole number, or maybe a fraction with small numerator and denominator. If this does not work, try using π^2 instead. If you are using a machine to help you, discuss how you would plan your calculations if you had only your brain, a pen or pencil, and paper, like Euler. Speculate further about what Euler may have considered while doing all this, and why.

Exercise 1.17. Verify that Euler’s summation formula specializes to the formulas of Bernoulli for sums of powers. Explain what the constant C is; pay special attention to the final terms of Bernoulli’s formulas.

Exercise 1.18. Verify that if we use trapezoids instead of rectangles to approximate the area represented by $\int_c^n f(x)dx$, we obtain the trapezoid rule: $\sum_{i=c+1}^n f(i) - \left(\frac{f(n)-f(c)}{2}\right) \approx \int_c^n f(x)dx$.

1.2 Archimedes Sums Squares to Find the Area Inside a Spiral

In 216 B.C.E., the Sicilian city of Syracuse allied itself with Carthage during the second Punic war, and thus was attacked by Rome, portending what would ultimately happen to the entire Hellenic world. During a long siege, soldiers of the Roman general Marcellus were terrified by the ingenious war machines defending the city, invented by the Syracusan Archimedes (c. 287–212 B.C.E.). These included catapults to hurl great stones, as well as ropes, pulleys, and hooks to raise and smash Marcellus’s ships, and perhaps even burning mirrors setting fire to their sails. Finally though, probably through betrayal, Roman soldiers entered the city in 212 B.C.E., with orders from Marcellus to capture Archimedes alive. Plutarch relates that “as fate would have it, he was intent on working out some problem with a diagram and, having fixed his mind and his eyes alike on his investigation, he never noticed the incursion of the Romans nor the capture of the city. And when a soldier came up to him suddenly and bade him follow to Marcellus, he refused to do so until he had worked out his problem to a demonstration; whereat the soldier was so enraged that he drew his sword and slew him” [133, p. 97].

Despite the great success of Archimedes’ military inventions, Plutarch says that “He would not deign to leave behind him any commentary or writing on such subjects; but, repudiating as sordid and ignoble the whole trade of engineering, and every sort of art that lends itself to mere use and profit,