3

Curvature and the Notion of Space

3.1 Introduction

On June 10, 1854, at the University of Göttingen, in a lecture that nearly did not occur, Georg Friedrich Bernhard Riemann (1826–1866) proposed a visionary concept for the study of space [223, pp. 132–133]. To obtain the position of an unsalaried lecturer (Privatdozent) in the German university system, Riemann was required to submit an inaugural paper (Habilitationsschrift) as well as to present an inaugural lecture (Habilitationsvortrag). The topic of the lecture was selected from a list of three provided by the candidate, with tradition suggesting that the first would be chosen. The most prominent member of the faculty at Göttingen and arguably the preeminent mathematician of his time, Carl Friedrich Gauss (1777–1855) passed over Riemann's first two topics (concerning his recent investigations into complex functions and trigonometric series), and chose the third as the subject of the lecture: *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (On the Hypotheses That Lie at the Foundations of Geometry) [173, p. 22]. Gauss's decision, undoubtedly motivated by his own unpublished work on non-Euclidean geometry, elicited a lecture that changed the course of differential geometry. Some years prior to Riemann's lecture, Gauss had developed a consistent system of geometry in which the Euclidean parallel postulate (see below) does not hold, but wishing to avoid controversy, he did not publish these results [101]. Riemann, however, did not present a lecture tied to the tenets of a particular geometry (Euclidean, hyperbolic, or otherwise), but offered a new paradigm for the study of mathematical space with his notion of an n-dimensional manifold. His ideas remain the standard for the classification of space today. Although many modern textbooks on geometry and topology offer a rather technical definition of a manifold, the ultimate goal of this chapter is to present Riemann's own lucid description of what space ought to be. What developments in mathematics helped to precipitate Riemann's lecture? What mathematical concepts are needed for an appreciation of the ideas therein? Why have his thoughts endured the test of time?

The title of Riemann's lecture, *On the Hypotheses That Lie at the Foundations of Geometry* [193], suggests immediately that the essay concerns fundamental principles (axioms) of geometry, which the author regards not as given, as in most treatises on the subject, but as the hypotheses of an empirical science. This is indeed the case, with the key axiom in question being Euclid's fifth postulate, the parallel postulate, which in modern parlance states, Given a line L and a point P not on L, then there is a unique line M through P parallel to L. For nearly two millennia mathematicians and philosophers had tried to prove the parallel postulate from Euclid's first four axioms, an activity that reflected the fundamental belief that space is Euclidean, and that this fact must follow logically from more basic ideas of geometry.

In his *Critique of Pure Reason* (1781), Immanuel Kant (1724–1804) espouses the idea of Euclidean geometry as a philosophical necessity. Compare the following two viewpoints, the first by Kant, the second by Riemann:

> Space is not a conception which has been derived from outward experiences. ... the representation of space must already exist as a foundation. Consequently, the representation of space cannot be borrowed from the relations of external phenomena through experience [129, p. 23].

> Thus arises the problem of seeking out the simplest data from which the metric relations of Space can be determined ... the most important system is that laid down as a foundation of geometry by Euclid. These data are—like all data—not necessary, but only of empirical certainty, they are hypotheses ... [166, p. 269].

Riemann's use of "metric relations" above refers to the determination of the length of a segment or an arc, which is then used to determine the nature of space. This stands in direct opposition to Kant's "the representation of space cannot be borrowed from the relations of external phenomena through experience."

Riemann's treatment of space does not involve a study of the axioms of geometry, but instead the inauguration of a new concept for thinking about space. A detailed study of the axiomatic geometry that results from replacing the parallel postulate by a particular case of its negation was undertaken by János Bolyai (1802–1860), Nikolai Lobachevsky (1792–1856), Gauss, and others, for which the reader is referred to [17, 150, 198]. Although pioneering in its spirit, the bold new geometry of Bolyai and Lobachevsky, today called hyperbolic geometry, suffered from a key drawback: neither author provided an example of hyperbolic geometry. Riemann's notion of a manifold offers a setting that encompasses not only Euclidean and hyperbolic geometry, but also many other new geometries, and proved essential for the study of relativity and space-time in the work of Albert Einstein (1879–1955) [58] and Hermann Minkowski (1864–1909) [172].

As the reader will discover, a manifold, in Riemann's words, is a continuous transition of an instance, and need not be contained in two- or even three-dimensional Euclidean space. Moreover, manifolds may have any dimension, finite or infinite. The crucial feature of manifold theory that allows non-Euclidean geometries is curvature. Whereas the Euclidean plane is flat, the surface of a sphere or the saddle $z = x^2 - y^2$ are both curved and provide two examples of manifolds. To determine what alternative to the parallel postulate holds on a curved space, the idea of line must be generalized to an arc of shortest distance between two points (a geodesic). On a sphere, such arcs form great circles, and given a great circle L and a point P not on L, there are no great circles through P parallel to L. In short, there are no parallel "lines" on a sphere, where "line" must be interpreted as a great circle.

Spheres have constant positive curvature and provide a setting for what is known today as elliptic geometry. A surface of constant negative curvature (Exercise 3.26) is a model for hyperbolic geometry, where given a "line" L and a point P not on L, there are many "lines" through P parallel to L. Notice how the determination of "lines" on a surface (two-dimensional manifold) provides the proper version of the parallel postulate for that surface. Of course, there must be a method to determine arc length on a surface (or within a manifold) in order to identify the geodesics. "Thus," as Riemann states "arises the problem of seeking out the simplest data from which the metric relations of Space can be determined ... ." These metric relations, as the visionary genius claims, are determined by the curvature of the manifold.

What is curvature and how is it computed? Through a sequence of selected original sources, answers to these questions will be provided. The goal of the chapter is not to conclude with what today is called the Riemann curvature tensor, an advanced topic [166, 223], but to tell the story of curvature through the work of pioneers such as Christiaan Huygens (1629–1695), Isaac Newton (1642–1727), Leonhard Euler (1707–1783), and Carl Friedrich Gauss (1777–1855). For a surface, Riemann's notion of curvature is simply Gaussian curvature, and to discuss the curvature of higher-dimensional manifolds, Riemann considers the Gaussian curvature of certain two-dimensional surfaces within the manifold.

Although the story of curvature could begin with the work of Apollonius (250–175 B.C.E.) on normals to a plane curve and the envelope such normals form when drawn to a conic section [5], the goal of Apollonius's *Conics* appears to be the study and applications of conic sections and not the description of how a plane curve is bending. To construct the envelope of a plane curve at the point P, consider another point P' on the curve very close to P and draw perpendiculars to the curve through the points P and P'. Suppose that the two perpendiculars intersect at the point Q on one side of the curve. The limiting position of Q as P' approaches P is designated as a point on the envelope, with the envelope itself being the set of all such limiting points as different locations are chosen for P to begin the process. The envelope to a general plane curve, not just a conic section, was systematically studied by Huygens in his work on pendulum clocks, and such an envelope he called an evolute [256] (see below). The curvature of a given curve at some point P

is the reciprocal of the length of the normal drawn from P to the evolute. Huygens's construction is general enough that it can be applied to any curve in the plane (with a continuous second derivative). His method, taken from the *Horologium Oscillatorium* (Pendulum Clock) (1673) [121] is presented in Section two.

The *Horologium Oscillatorium* finds its motivation in a rather applied problem: the need, during the Age of Exploration, for ships to determine longitude when navigating the round earth [221]. If a perfect timekeeper could be built, then longitude could be determined at sea by first setting the clock to read noon when the sun is at its highest point at the port of embarkment. A reading of the clock at sea when the sun is again at its highest point would yield a discrepancy from noon, depending on how many degrees of longitude the ship had progressed from embarkment, with one hour corresponding to $360^\circ/24 = 15^\circ$. Although in 1659 Huygens did construct a chronometer that theoretically keeps perfect time, his device did not perform reliably on the high seas [256]. The history of science credits the Dutch inventor with constructing the first working pendulum clock,[1] while physics is indebted to Huygens for the isochronous pendulum, a special type of pendulum with the mathematical property to keep perfect time. So acute was the need to determine longitude at sea that the British government issued the Longitude Act on July 8, 1714, which offered £20,000 for a method to determine longitude to an accuracy of half a degree of a great circle. The prize, after much haggling with the Board of Longitude, was awarded nearly in full to John Harrison (1693–1776) for his maritime clock known as H-4 [22, 221].

Huygens's isochronous pendulum, although it did not solve the longitude problem, employs certain techniques that soon became standard for the study of curvature of plane curves. The first of these is the osculating circle, and the second is the radius of curvature. Given a curve in the plane, to find its curvature at some point B, construct a circle that best matches the curve at B. This is the osculating circle; its radius is the radius of curvature at B, and the measure of curvature is the reciprocal of the radius. The locus of the centers of the osculating circles as the point B moves along the given curve forms what Huygens calls the evolute of the original curve. The reader is invited to witness how the ideas of the osculating circle and radius of curvature arise in the original work of Huygens presented in Section two. Huygens's scientific legacy as portrayed in modern physics texts is touched on in Exercise 3.8.

Astonishingly, Huygens arrived at his description of the radius of curvature before the development of the differential or integral calculus. His results are stated in geometric terms without the use of derivatives or even equations. Moreover, the term osculating circle was coined by Gottfried Wilhelm Leibniz (1646–1716), who spent the years 1672–1676 in Paris, where he met the renowned Huygens and received a copy of the *Horologium Oscillatorium*

[1] The idea for the use of a pendulum as a regulating device in a clock goes back to Galileo, but he never built such a clock [221, p. 37].

[117]. An analytic expression for the radius of curvature was found by Isaac Newton (1642–1727) and appears in his *De methodis serierum et fluxionum* (Methods of Series and Fluxions), published in 1736. The reading selection of Section three is precisely Newton's solution to what he states as "To find the curvature of any given curve at a given point" [178, p. 150]. With his newly developed calculus of fluxions (differential calculus), Newton arrived at an equation for curvature that can easily be implemented and is equivalent to expressions for curvature found in modern calculus texts. The geometry behind Newton's construction, however, is strikingly similar to that of Huygens. It is profitable to compare Huygens's construction of the evolute in Figure 3.5 with Newton's derivation of the radius of curvature in Figure 3.8. Newton has essentially assigned coordinates and their associated fluxions to the geometry behind the osculating circle.

This concludes the discussion on the curvature of plane curves, which is necessary for an understanding of the curvature of other objects. Curves in three-dimensional (Euclidean) space had been studied by Alexis Clairaut (1713–1765) [230, p. 100], and described as curves of double curvature in his 1731 text *Recherches sur les courbes à double courbure* (Researches on curves of double curvature), a topic not pursued here. See [223, p. 38] for further details. The chapter instead moves forward with the study of surfaces (two-dimensional manifolds) in Euclidean three-space with emphasis on their curvature. In this regard, we turn to the contributions of the prolific Leonhard Euler (1707–1783).

In a paper presented to the St. Petersburg Academy of Science in 1775, *De repraesentatione superficiei sphaericae super plano* (On Representations of a Spherical Surface on the Plane) [66, v. 28, pp. 248–275], Euler proved what cartographers had long suspected, namely the impossibility of constructing a flat map of the round world so that all distances on the globe are proportional (by the same constant of proportionality) to the corresponding distances on the map. In a preceding paper (1770), *De solidis quorum superficiem in planum explicare licet* (On Solids Whose Surfaces Can Be Developed in the Plane) [66, v. 28, pp. 161–186], Euler had studied the problem of describing all surfaces that can be mapped to the plane. In doing so he introduced two techniques that would become standard tools in differential geometry. The first is the use of two parameters to describe points on the surface, an idea used again by Gauss and extended by Riemann to higher-dimensional manifolds. The second is the use of a line element, i.e., the "metric data" needed to compute arc length on a surface. Gauss would later re-prove Euler's result on map projections [84] as a special case of a more general theorem in which the problem of mapping one surface onto another (not necessarily a plane) is reduced to knowing the curvature of both surfaces. If a distance-preserving map between two surfaces exists, then both surfaces must have the same value of

Gaussian curvature at corresponding points, a theorem that Gauss christens the *theorema egregium*[2] (remarkable theorem).

How exactly is the curvature of surfaces computed? The reading selection of Section four offers Euler's answer to this question from his 1760 essay *Recherches sur la courbure des surfaces* (Researches on the Curvature of Surfaces) [66, v. 28, pp. 1–22]. He begins by considering a planar cross section of the surface, and then determines the curvature of the curve formed by the intersection of the plane with the surface. At a given point P of the surface, Euler further restricts his attention to those planes that are perpendicular to the surface, and identifies two "principal" cross sections at P, one with maximum curvature and one with minimum curvature. Moreover, any other perpendicular cross section at P has a value for its curvature that can be expressed in terms of these maximum and minimum values via a simple formula. In this way Euler reduces the curvature of surfaces to that of curves. The Euler Archive [136, Eneström 333] offers an English translation of his original proof of this result.

The calculational genius begins his paper thus (translated from the original French):

> In order to know the curvature of curved lines, the determination of the radius of the osculating circle offers the proper method .... But for ... surfaces, one would not even know how to compare the curvature of the surface with that of a sphere, as one can always compare the curvature of a curved line with that of a circle [66, v. 28, p. 1].

The idea expressed here, that the curvature of a surface might be computed in terms of an osculating sphere, much as the curvature of a plane curve is expressed in terms of an osculating circle, is not realized. (See the conclusion of Exercise 3.17 for the quadratic surface that best matches a given surface at a given point.) Carl Friedrich Gauss, however, does make incisive use of an auxiliary sphere to compute the curvature of surfaces (see below), although this sphere is not, strictly speaking, the two-dimensional analogue of the osculating circle. Foreshadowing Gauss's deep results, Oline Rodrigues (1794–1851) [230, p. 116] had studied the ratio of a small area on a surface and the corresponding area on an auxiliary sphere [197], but did not develop this idea to the extent of Gauss. Furthermore, Sophie Germain (1776–1831) had introduced the notion of a referent sphere to a surface and proposed that this sphere have curvature given by the mean (average) of the maximum and minimum cross-sectional curvatures found by Euler [88, 89]. She does not, however, offer a construction that would show how the referent sphere would arise out of geometric considerations (such as the construction of an osculating circle). Nonetheless, Germain's mean curvature proved to be essential in her work on elasticity [24], and later became a key tool in the study of minimal surfaces (surfaces of least area with prescribed boundary [224]).

[2] The converse of the *theorema egregium*, at least for surfaces of constant curvature, was studied by Ferdinand Minding (1806–1885) [170].

In a deep and highly polished essay, *Disquisitiones generales circa superficies curvas* (General Investigations of Curved Surfaces) (1827) [84], Carl Friedrich Gauss (1777–1855) introduced his own concept for the measure of curvature of a surface at a point. Unlike Euler, who had considered planar cross sections to a surface, Gauss begins by considering vectors normal (perpendicular) to the surface, and then transports these vectors to an auxiliary sphere. By making an adroit comparison between the area of a small triangle on the surface around some point P and the corresponding area of the triangle on the auxiliary sphere, Gauss develops a formula for the curvature of the surface at P in terms of a single value. In an elegant and unifying theorem, the preeminent geometer proves that this value, today known as Gaussian curvature, is equal to the product of the maximum and minimum cross-sectional curvatures found by Euler. The essay continues with a proof of the *theorema egregium*, that one surface can be mapped onto another in a distance-preserving fashion only if the surfaces have the same value of Gaussian curvature at corresponding points. This is followed by a study of geodesics (arcs of shortest length) on surfaces, and a detailed analysis of angles in a geodesic triangle (a triangle, all sides of which are geodesics).

In Euclidean geometry all triangles have angle sum 180°, a result that is itself logically equivalent to the parallel postulate. Gauss proves, however, that on a surface of negative curvature, a geodesic triangle has angle sum less than 180°, and on a surface of positive curvature, a geodesic triangle has angle sum greater than 180°. In particular, “The excess of the angles of a triangle formed by shortest paths over two right angles is equal to the total curvature of the triangle” [51, p. 90]. The total curvature here refers to the integral of the Gaussian curvature over the triangle. Notice how vividly curvature enters into a result that transcends a basic tenet of Euclidean geometry. The curvature of the Euclidean plane is, of course, zero. Gauss's own unpublished work on hyperbolic geometry served in part to motivate these results [51]. Before the authorship of *Disquisitiones superficies* or even its 1825 draft, the German master had been fully aware of the logical basis for a geometry satisfying Euclid's first four axioms, but not the fifth, yet he shared with few his work on non-Euclidean geometry [17, 101]. Primarily to avoid controversy, Gauss did not publish his results on hyperbolic geometry, and in a letter concerning János Bolyai's discovery in this field, Gauss wrote to János's father on March 6, 1832: “My intention was, in regard to my own work [on non-Euclidean geometry] of which very little up to the present has been published, not to allow it to become known during my lifetime” [249, p. 52]. Nowhere in *Disquisitiones superficies* is there mention of the parallel postulate.

Further inspiration for *Disquisitiones superficies* may have been drawn from Gauss's own field work as surveyor of the Kingdom of Hanover (now a German state) during the years 1821–1825 [51, p. 129]. Following his geodetic survey during the summer of 1825, he writes to a friend Christian Schumacher on November 21 of that year:

> Recently I have taken up again a part of the general investigations on curved surfaces which are to form the basis of my projected essay on advanced geodesy. It is a subject which is as rich as it is difficult, and it takes me from accomplishing anything else [85, p. 400].

The reading selection from *Disquisitiones superficies* in Section five includes a derivation for Gaussian curvature of a surface in terms of its partial derivatives, as well as the description of a surface in terms of two parameters p, q. The curious reader is encouraged to consult Gauss's original tract [87] for a proof of the *theorema egregium*, which relies on the equations for the "metric data" (identified as E, F, G in Gauss's notation) needed to compute the length of curves in terms of the two-parameter coordinate system. The proof is too lengthy to be reproduced in this chapter.

The publication of *Disquisitiones superficies* precipitated the study of surfaces with specific properties that could then be stated in terms of Gaussian curvature, such as surfaces of constant curvature [192]. In addition to this the search for minimal surfaces entered a new era of growth around 1830. Before then the only (nontrivial) examples of minimal surfaces had been discovered by Euler and Jean-Baptiste Meusnier (1754–1793), namely the catenoid and the helicoid[3] [192]. The final section of the chapter, however, moves forward with Riemann's essay *On the Hypotheses That Lie at the Foundations of Geometry*, where the notion of manifold offers a new paradigm for the study of space. Generalizing from the idea of a curved surface, Riemann's notion of extended quantity encompasses objects of any dimension, and, moreover, objects that do not necessarily exist in a three-dimensional Euclidean world. (For a two-dimensional surface that cannot be constructed in an ambient three-dimensional world, see Exercise 3.31.) Riemann clearly intended that the metric data needed to compute the length of a curve in a manifold be given by generalizing the ideas of Gauss to higher dimensions. In a striking result, the visionary Riemann claims that the curvature of the manifold determines the metric: "If the curvature is given in $\frac{1}{2}\,n(n-1)$ surface directions at every point, then the metric relations of the manifold may be determined" [166, p. 274]. The surfaces referred to here are certain two-dimensional surfaces within the n-dimensional manifold, and curvature refers to the Gaussian curvature of these surfaces. Thus, curvature determines the metric, which in turn determines the geodesics on the manifold, and these provide us with the version of the parallel postulate that holds in a given manifold. In this sense, curvature determines the nature of space.

Riemann's lecture was meant for a general scholarly audience, and as such contains virtually no formulas for the metric relations or curvature. In a subsequent paper (1861) [194, pp. 391–404] he does introduce specific formulas for these, although it is not the goal of the chapter to present this material. All of the claims in his 1854 lecture can be substantiated; for proofs, the reader is referred to the specialized texts [166, 223]. In the one and one-half

[3] The mathematical properties of these surfaces are discussed in [224].

centuries following his ground-breaking address, Riemann's ideas have proven very fertile, with an entire subject, Riemannian geometry [50, 199], having its roots in this single source. In the twentieth-century work of Einstein and Minkowski, manifold theory proved to be essential in the study of relativity and four-dimensional space-time [58, 172].

Since Riemann and Einstein, the classification of manifolds has continued in all dimensions, with recent spectacular progress in the fourth dimension [188]. In particular, by the work of Simon Donaldson (1957–), the notion of derivative has special interpretations in dimension four that do not occur in any other dimension [9, pp. 3–6]. The most elusive dimension, however, remains the third, with an outstanding problem being the classification of three-dimensional manifolds that can be continuously deformed into the three-dimensional sphere:

$$S^3 = \left\{ (x_1, x_2, x_3, x_4) \in \mathbf{R}^4 \mid \sum_{i=1}^{4} x_i^2 = 1 \right\}.$$

This problem, known as the three-dimensional Poincaré conjecture, has been solved for every dimension except possibly the third, with even the herculean task of dimension three apparently also solved as this text goes into print. The four-dimensional Poincaré conjecture was only recently proved (in 1982) by Michael Freedman (1951–) [168, pp. 13–15].

Our story of curvature bears witness to some very applied problems, such as constructing an accurate timekeeper, and mapping the round globe onto a flat plane, whose solutions or attempted solutions resulted in key concepts for an understanding of space. The development of relativity and space-time provided a particular impetus for the study of manifolds, with the curvature of space-time being a key feature that distinguishes it from Euclidean space. The reader is invited to see [181] for a delightful informal discussion of curvature, and [222, 223] for a more advanced description of manifold theory. This chapter closes with the open problem of the classification of space, space in the name of mathematics and philosophy, space in the name of physics and astronomy, space in the name of curiosity and the imagination.

3.2 Huygens Discovers the Isochrone

Holland during the seventeenth century was a center of culture, art, trade, and religious tolerance, nurturing the likes of Harmenszoon van Rijn Rembrandt (1606–1669), Johannes Vermeer (1632–1675), Benedict de Spinoza (1632–1677), and René Descartes (1596–1650). Moreover, the country was the premier center of book publishing in Europe during this time, with printing presses in Amsterdam, Rotterdam, Leiden, the Hague, and Utrecht, all publishing in various languages, classical and contemporary [108, p. 88]. Into this environment was born Christiaan Huygens (1629–1695), son of a prominent statesman and diplomat.