TEACHING DISCRETE MATHEMATICS ENTIRELY FROM PRIMARY HISTORICAL SOURCES

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Abstract: We describe teaching an introductory discrete mathematics course entirely from student projects based on primary historical sources. We present case studies of four projects that cover the content of a one-semester course, and mention various other courses that we have taught with primary source projects.

Keywords: primary historical sources, original historical sources, pedagogy, discrete mathematics

1 INTRODUCTION

In [7] we described teaching undergraduate mathematics using student projects based on primary historical sources. We presented our motivation, our pedagogy, the advantages and challenges of teaching with primary historical sources, student responses, and three sample project case studies. All our projects and how we have used them in various courses may be found in [2, 3, 4, 5, 23].

Our student projects are modular, and can be used to replace even a small part of the textbook material in a course. Among the myriad positive student comments about learning mathematics from primary sources, remarks like the following inspired us to aim higher, for more than just small parts of a course:

> You don't learn to play an instrument without listening to those who played it best.

It gives you the sense of how math was formed which prepares you for how to think up new, innovative mathematics for the future.

It puts the ideas in context where textbooks fail.

Textbooks are like inventions without instruction manuals.

In this sequel paper to [7], we describe and analyze how we went further, developing enough project material to eventually teach several standard one-semester courses entirely from our projects, jettisoning standard textbooks. The courses we have taught entirely from projects based on primary historical sources are lower division discrete mathematics and upper division combinatorics and mathematical logic. Here we will present four projects that we have used in multiple ways to teach an entire introductory discrete mathematics course.

As we built modular projects into existing courses, and eventually to fill entire courses, we evolved a particular guided reading approach common to all our student projects. With our own writing we guide the student in their own reading and study of selected primary source excerpts. A key feature is that we interrupt the primary source with clarifying comments, and with exercises. The exercises often have an intentional didactic purpose, provoking students to gain through their own intellectual work specific critical insights and understanding from their guided engagement with key spots in the primary source. This enables better understanding of both the source and the mathematics for going forward. The exercises may address aspects of rigor and proof, reflection on the evolution of mathematical processes, of terminology, notation and definitions, and other components of developing mathematical competency that can emerge from guided sense-making of a primary source. Our projects also have a number of other design features discussed at length in [6, 7, 8].

While all our guided reading projects have pedagogical commonalities as detailed above, they also have substantial individual differences, particularly in the choices made for how to incorporate primary source material. The reader may look for these as we present the four projects featured below. In the Conclusion we will discuss the nature of the differences between these four projects.

2 FOUR CASE STUDY STUDENT PROJECTS IN DISCRETE MATHEMATICS WITH PRIMARY SOURCES

We have taught an entire one-semester introductory discrete mathematics course from primary source projects. Standard topics in such a course often include basic logic, sets, functions, relations, elementary number theory, and proof techniques with special emphasis on mathematical induction. The following four of our projects fit well with these topics:

1. "Deduction through the Ages: A History of Truth," with sources by Chrysippus, Boole, Frege, Russell and Whitehead, Wittgenstein, and Post.
2. "An Introduction to Symbolic Logic," with sources by Russell and Whitehead.
3. "An Introduction to Elementary Set Theory," with sources by Cantor and Dedekind.
4. "Pascal's Treatise on the Arithmetical Triangle: Mathematical Induction, Combinations, the Binomial Theorem and Fermat's Theorem."

We have used various combinations of these projects to teach a 3-credit 15-week one-semester course. The course has three basic units: logic, set theory and functions, and mathematical induction. Each unit occupies roughly four to five weeks of the course.

The logic unit consists of basic propositional logic and quantification. This is covered by either the first project and the second half of the second project, or the entirety of the second project. The unit on sets and functions is covered by the third project, and the unit on mathematical induction is the main thrust of the fourth project, which also includes optional number theory. We usually begin with the logic unit, but we have taught the course with the latter two projects in either order. Each project has detailed notes to the instructor, which outline different possible uses of the project in and out of the classroom. In the subsequent four subsections we will present overviews, topical highlights, and sample primary source excerpts and student exercises from these four projects. Each subsection title consists of the italicized title of one of the four projects, along with an indication of some of the key topics featured.

We have also developed and used several additional projects in teaching discrete mathematics. In the Conclusion we will discuss how these may be combined to create alternative course syllabi.

2.1 *Deduction through the Ages: A History of Truth*: Rules of Deduction, Logical Implication, Propositional Logic and Truth Tables

Almost every introductory discrete mathematics course covers elementary propositional logic, including the truth table for an implication (an "if-then" statement), often written $p \to q$, read "p implies q." The truth values of an implication when the hypothesis p is false are counterintuitive and often a stumbling block for students, not to mention instructors, who struggle to justify these truth values with strained examples of implications, suggesting that truth tables are sometimes verifiable by mere example, while at other times requiring rigorous computation. The status of a logical implication in today's textbooks is that of a definition, although the intellectual struggle leading up to the common acceptance of this definition is almost never discussed in modern textbooks, and certainly not from the history of deductive thought. While such a history could encompass many possible sources, the teaching project "Deduction Through the Ages: A History of Truth" [21] offers excerpts from a few selected sources from Antiquity to the twentieth century that illuminate and focus how deductive reasoning has been reduced to the truth values of one type of statement, the logical implication.

The project opens with five rules of deduction stated verbally by the ancient Greek philosopher Chrysippus (ca. 280–206 B.C.E.), which for the reader's convenience are reproduced below [17, p. 189] [22, p. 212–213].

1. If the first, [then] the second. The first. Therefore, the second.
2. If the first, [then] the second. Not the second. Therefore, not the first.
3. Not both the first and the second. The first. Therefore, not the second.
4. Either the first or the second. The first. Therefore, not the second.
5. Either the first or the second. Not the first. Therefore, the second.

The project offers examples of these rules, but more importantly, asks the student why five separate rules are necessary and which rules, if any, are equivalent. Insight into the possible equivalence of certain rules is gained by reading selected historical sources, and forms the leitmotif of the project. Verbal argument asserting the equivalence of Chrysippus's rules is difficult and a more symbolic method of discourse is sought.

The next selection in the project is from George Boole's (1815–1864) opus *An Investigation of the Laws of Thought* [11, 12], where we witness an initial attempt to reduce language to a symbolic format. Boole borrows the symbols $\times$, $+$, $-$ form algebra, roughly denoting "and," "or" and "not" respectively, although the symbols no longer maintain all of their algebraic proprieties, but satisfy "the laws of thought." With this source, progress is made in the reduction of verbal argument to calculation, although the equivalence of Chrysippus's rules remains open.

An entirely different system of notation, the "Begriffsschrift" or *concept-script* is introduced in the work of Gottlob Frege (1848–1925) [18, 19]. The center piece of his notation is the condition stroke, written

$$\begin{array}{l} \top\, \xi \\ \llcorner\, \zeta\,, \end{array}$$

whose truth value depends on the truth values of the individual arguments ζ and ξ. The entire symbol is true, except when it begins with a true statement (ζ) and ends with a false statement (ξ). In this case only is the condition stroke false. This compares well with Philo of Megara's (ca. 4th century B.C.E.) verbal statement that a valid hypothetical proposition is "that which does not begin with a truth and end with a falsehood" [29, II. 110]. Surprisingly four of Chrysippus's rules can be written in terms of the condition stroke by asserting or negating the stroke or certain of its arguments.

The goal of reading the Frege selection is not the mastery of a strange and new notation, but the insight this symbol lends to the possible equivalence of Chrysippus's rules. Four separate rules can be written in terms of one root symbol, the condition stroke. The reader is encouraged to consult the actual project [21] for how this happens. Student engagement with the Frege reading was high. Students were able to formulate questions about the meaning of logical connectives such as "and," "or" and "not" in terms of symbolic constructions involving the condition stroke, a first step in bridging the gap between verbal and symbolic arguments.

Although Frege does not offer a formal mathematical definition of an implication, this is pursued by the twentieth century writers Bertrand Russell (1872–1970) and Alfred North Whitehead (1861–1947) in their monumental *Principia Mathematica* [27] . Additionally, Russell and Whitehead introduce their own notation for "and," "or" and "not" similar to the notation found in present-day textbooks. We read that "and" is denoted by "$\cdot$," (inclusive) "or" is denoted "$\vee$" and "not" is denoted "$\sim$." With this in hand, Russell and Whitehead define an implication in terms of a statement of equivalent deductive power involving an (inclusive) "or" statement similar to that used in Chrysippus's fifth rule. Russell and Whitehead write [28, p. 7–8]:

The Implicative Function is a propositional function with two arguments p and q, and is the proposition that either not-p or q is true,

> that is, it is the proposition $\sim p \vee q$. Thus, if p is true, $\sim p$ is false, and accordingly the only alternative left by the proposition $\sim p \vee q$ is that q is true. In other words if p and $\sim p \vee q$ are both true, then q is true. In this sense the proposition $\sim p \vee q$ will be quoted as stating that p implies q. The idea contained in this propositional function is so important that it requires a symbolism which with direct simplicity represents the proposition .... The symbol employed for "p implies q", i.e. for "$\sim p \vee q$" is "$p \supset q$." This symbol may also be read "if p, then q." ...

Student exercises include writing (most of) Chrysippus's rules in terms of the implication symbol of Russell and Whitehead.

In 1921 both Ludwig Wittgenstein's (1889–1951) *Logische-philosophische Abhandlung* [30, 31] and Emil Post's (1897–1954) "Introduction to a General Theory of Elementary Propositions" [26] appeared in print. Both authors introduce a tabular format to display the truth values of propositions. Wittgenstein calls his tables "schemata," while Post uses the term "truth tables," which is in current use. Building on Russell and Whitehead's definition of an implication, the truth table of "p implies q" ($p \supset q$) follows easily. A definitive answer to the equivalence of certain of Chrysippus's rules follows now from calculations with truth tables, forming concluding exercises for students (and the instructor).

When taught solely from a standard textbook, students are willing to memorize the truth values of an implication, certainly if required for an examination. The main pedagogical goal of learning from primary historical sources is to provide students with an understanding of the ideas behind the present mathematical paradigm and offer insight into how and why this paradigm evolved into its present form. Students engage with each source by reading short excerpts, answering questions about the meaning of a passage or performing deductions using the methods set forth in a given source [21]. In-class activities include discussion of certain exercises with various students presenting separate steps of the solution on the blackboard.

At the conclusion of this project, students were asked to write a paper detailing the evolution of the thought process that has found resolution in the modern truth table for an implication. This is an activity not to be found in a modern textbook. As a guide to the paper, students were asked to trace the development of deductive thought by examining how just one of Chrysippus's rules can be recast in terms of each of the major sources in this project. For example, the following exercises (taken from [21]) concerning his third rule of inference could form the core of a paper:

Exercise 1. The major premise of the third rule of inference is the negation (not the case) of an "and" statement. Rewrite "It is not the case both that it is day and it is night" as an "if-then" statement so that when the pattern of the first rule of inference is followed, the minor premise becomes "It is day," and the conclusion becomes "it is not night."

Exercise 2. Consider the example of Chrysippus's third rule of inference: "It is not the case both that it is day and it is night. It is day. Therefore, it is not night." Let ζ denote the proposition "It is day," and let ξ denote the proposition "It is night." Write the major premise of the third rule entirely in Frege's notation. Write the minor premise and conclusion using his notation as well. Be sure to justify your answer.

Exercise 3. (a) Rewrite the major premise from the example of Chrysippus's third rule: "It is not the case both that it is day and it is night" in the notation of *Principia Mathematica*. Begin with two elementary propositions denoting "It is day" and "It is night." Rewrite the major premise as an "inclusive or" statement using a rule of logic discovered in this section. Find an implication ("if-then" statement) that is logically equivalent to the major premise.

(b) Can the negation of every "and" statement be written as an implication? Justify your answer. Conversely, can every implication be written as the negation of an "and" statement? Justify your answer in this case.

Exercise 4. Letting p denote the elementary proposition "It is day," and q denote "It is night," write the compound statement "It is not the case both that it is day and it is night" in the notation of *Principia Mathematica.* Construct the schema or truth table for this compound statement as well. Find an implication ("if-then" statement) involving p and q that has an identical schema or truth table, and justify your answer.

Student comments about learning from the historical sources in this project include the following:

> It is neat to see how each new generation of logicians built on previous work from former logicians, and how the work is all interconnected.
>
> Someone who is studying the Russell and Whitehead notation can easily retrace and write the statement as an equivalent 'if-then' statement using one of Chrysippus's rules.
>
> The use of Frege notation really clarifies the relationship of the rules of logic and deductive thinking.
>
> The final mathematician that seems to effortlessly unite the verbal logic with the symbolic logic is Emil Post who developed the method we use now to represent the truth values of compound statements. His method of depicting propositional logic is a masterful display of how the development of mathematical logic has evolved.

2.2 *An Introduction to Symbolic Logic*: Universal and Existential Quantifiers

After covering the project "Deduction Through the Ages" on propositional logic, students have a good understanding of correct forms of reasoning, as well as how to calculate truth values of a compound proposition from truth values of elementary propositions of which it is composed. But elementary propositions could have a very complex structure of their own, which requires development of predicates and quantifiers. This we do in the second part of the project "An Introduction to Symbolic Logic" [9], which introduces students to the basics of symbolic logic based on Russell and Whitehead's magnum opus *Principia Mathematica* [27].

The second part of the project starts with analyzing the statement "every prime number greater than 2 is odd," which requires such new concepts as individual variables, predicates, their truth sets, and quantifiers. This is followed by reading Russell and Whitehead. Their source provides a very intuitive explanation of why it is necessary to assign a particular value to an individual variable x for a predicate involving x to make an assertion, and hence become a proposition [27, Vol. 1, p. 15]:

Propositional Functions. Let ϕx be a statement containing a variable x and such that it becomes a proposition when x is given any fixed determined meaning. Then ϕx is called a "propositional function"; it is not a proposition, since owing to the ambiguity of x it really makes no assertion at all. Thus "x is hurt" really makes no assertion at all, till we have settled who x is. Yet owing to the individuality retained by the ambiguous variable x, it is an ambiguous example from the collection of propositions arrived at by giving all possible determinations to x in "x is hurt" which yield a proposition, true or false. Also if "x is hurt" and "y is hurt" occur *in the same context*, where y is another variable, then according to the determinations given to x and y, they can be settled to be (possibly) the same propositions or (possibly) different propositions. But apart from some determination given to x and y, they retain in that context their ambiguous differentiation. Thus "x is hurt" is an ambiguous "value" of a propositional function.

This is followed by a number of exercises that provide students with a good understanding of individual variables and predicates, and how to calculate truth sets of predicates. Having this under their belt, students are ready to be introduced to the universal and existential quantifiers. This is done by reading the following excerpt from Principia [27, Vol. 1, pp. 15–16]:

> Thus corresponding to any propositional function...there is a range, or collection, of values, consisting of all the propositions (true or false) which can be obtained by giving every possible determination to x in ϕx...in respect to the truth or falsehood of propositions of this range three important cases must be noted and symbolised. These cases are given by three propositions of which one at least must be true. Either (1) all propositions of the range are true, or (2) some propositions of the range are true, or (3) no proposition of the range is true. The statement (1) is symbolised by "$(x).\phi x$," and (2) is symbolised by "$(\exists x).\phi x$." No definition is given of these two symbols, which accordingly embody two new primitive ideas in our system. The symbol "$(x).\phi x$" may be read "ϕx always," or "ϕx is always true," or "ϕx is true for all possible values of x." The symbol "$(\exists x).\phi x$" may be read "there exists an x for which ϕx is true"...and thus conforms to the natural form of the expression of thought.

This helps students appreciate how predicate logic builds on propositional logic, and usually leads to interesting in-class discussions about how the two relate to each other, and how the latter can be used in foundations of mathematics. In our experience, this provides a valuable enrichment of the content that is seldom addressed otherwise.

The project has a large body of carefully chosen exercises, which teach students to express many mathematical statements using quantifiers, as well as to help them understand a number of subtleties associated with the usage of quantifiers. In particular, special attention is payed to how the universal and existential quantifiers relate to each other through the negation.

The exercises form a particularly important part of the project, and were designed to simultaneously provide students with practice applying the ideas discussed and to extend the discussion to new material. Several exercises introduce concepts from logic, number theory, the theory of relations, and other topics that are not normally covered until later in a discrete mathematics course. For example, the following exercise gives students the first glimpse into equivalence relations and modular arithmetic, topics which are further elaborated in the set theory and mathematical induction units.

Exercise 1. For natural numbers x and y, define $x \texttt{ mod } y$ to be the remainder obtained upon dividing x by y. Allowing the variables x and y to range over the domain of natural numbers, let $P(x, y)$ denote the binary predicate "$x \texttt{ mod } 2 = y \texttt{ mod } 2$" and let $Q(x, y)$ denote the binary predicate "$x \texttt{ mod } 3 = y \texttt{ mod } 3$."

(a) Is $P(x, y)$ an equivalence relation? Justify your answer.

(b) Is $Q(x, y)$ an equivalence relation? Justify your answer.

(c) Fix a natural number n.

1. If $P(n, 2)$ holds, what must be true of n?
2. What about if $P(n, 1)$ holds?
3. For a fixed natural number n, must at least one of $P(n, 2)$ or $P(n, 1)$ hold?
4. Can both $P(n, 2)$ and $P(n, 1)$ hold for a single natural number n?

(d)
1. List the natural numbers n for which $Q(n, 1)$ holds, list the natural numbers n for which $Q(n, 2)$ holds, and list the natural numbers n for which $Q(n, 3)$ holds.
2. Do the three lists have any natural numbers in common? Does every natural number appear in at least one of the three lists?
3. For each of the lists, the collection of numbers appearing in that list is said to be an *equivalence class* of the equivalence relation $Q(x, y)$. What are the equivalence classes of the equivalence relation $P(x, y)$?

All such exercises are elementary, and may be taken as a stand-alone opportunity to study the primary material, or as an invitation to explore more advanced concepts. Because it is customary to cover logic at the beginning of a discrete mathematics course, these exercises could be used as a way of connecting logic to the material covered later in the course.

Several exercises are slightly open-ended. In our opinion, this stimulates independent thinking, as well as provides an opportunity for further in-class discussion. In our experience, such discussions enhance students' understanding of the material.

The concluding exercises of the project teach students how to reason with this new formalism, which is more powerful than the earlier introduced formalism of propositional logic, and how to utilize it in situations that can occur in everyday life.

2.3 *An Introduction to Elementary Set Theory*: Sets, Finite and Infinite, and their Comparison

Sets and their basic properties are usually covered in an introductory course on proofs. Our project "An Introduction to Elementary Set Theory" [10] offers students an introduction to sets and their basic properties based on original historical sources by Georg Cantor (1845–1918), the founder of set theory, and Richard Dedekind (1831–1916), Cantor's colleague and ally in promoting set theory.

The project opens with short biographical sketches of Cantor and Dedekind. The first part of the project introduces students to sets and such basic relations between sets as membership, subset, and equality relations. Students also become familiar with basic operations on sets such as union, intersection, set difference, Cartesian product, and powerset. The first part of the project ends with an informal discussion of Russell's paradox, which is usually rather popular among students.

The heart of the project is its second part, which discusses the "size" of a set, based on Cantor's and Dedekind's original ideas. Cantor compares the sizes of two sets as follows [13] [14, pp. 89–90] (note that in this old English translation sets are referred to as aggregates):

If for two aggregates M and N with the cardinal numbers $\mathfrak{a} = \overline{\overline{M}}$ and $\mathfrak{b} = \overline{\overline{N}}$, both the conditions:

(a) There is no part of M which is equivalent to N,

(b) There is a part N_1 of N, such that $N_1 \sim M$,

are fulfilled, it is obvious that these conditions still hold if in them M and N are replaced by two equivalent aggregates M' and N'. Thus they express a definite relation of the cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$ to one another...

We express the relation of $\mathfrak{a}$ to $\mathfrak{b}$ characterized by (a) and (b) by saying: $\mathfrak{a}$ is "less" than $\mathfrak{b}$ or $\mathfrak{b}$ is "greater" than $\mathfrak{a}$; in signs

$$\mathfrak{a} < \mathfrak{b} \text{ or } \mathfrak{b} > \mathfrak{a}.$$

Students are asked to verify that the relation $<$ is transitive ($\mathfrak{a} < \mathfrak{b}$ and $\mathfrak{b} < \mathfrak{c}$ imply $\mathfrak{a} < \mathfrak{c}$). They are also asked to define the relation $\leq$ and explore its basic properties. This naturally leads to discussing the Cantor-Bernstein theorem. While the rigorous proof of the theorem is beyond the scope of the project, this topic may lead to interesting in-class discussions.

Another related topic mentioned in the project is the law of trichotomy, that any two cardinal numbers are comparable; that is, for any two cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$, either $\mathfrak{a} < \mathfrak{b}$, $\mathfrak{a} = \mathfrak{b}$, or $\mathfrak{a} > \mathfrak{b}$. Of course, the rigorous treatment of the law of trichotomy and the Axiom of Choice is far beyond the scope of the project, but such conceptually nontrivial topics do stimulate rather interesting in-class discussions.

The project continues by comparing Cantor's and Dedekind's definitions of finite and infinite sets. The modern definition of finite and infinite sets is adopted from Cantor. On the other hand, in his famous original source *Was sind und was sollen die Zahlen?* [15] (for an English translation see [16]) Dedekind defines finite and infinite sets as follows [16, p. 63] (in Dedekind's usage 'system' means 'set' and 'similar' means 'equivalent'):

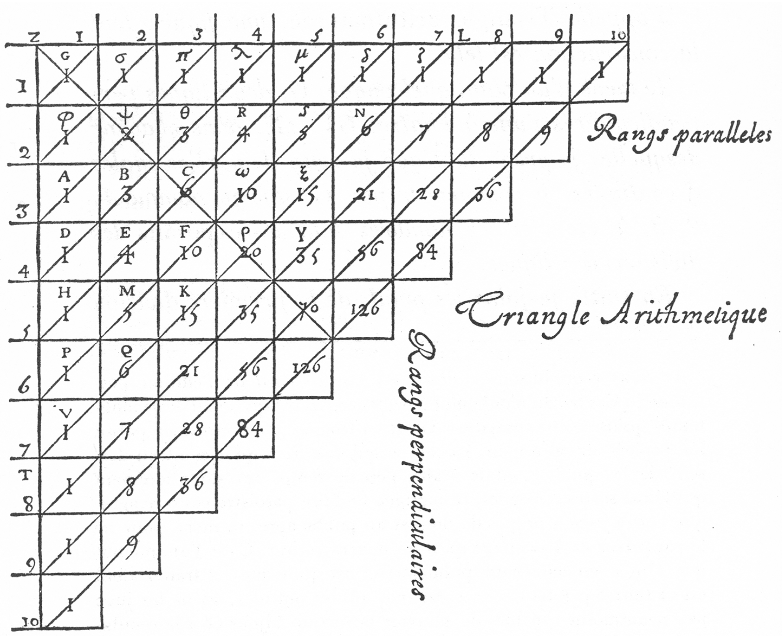
> A system S is said to be *infinite* when it is similar to a proper part of itself; in the contrary case S is said to be a *finite* system.

In the modern literature, sets that are infinite in the sense of Dedekind are called Dedekind infinite. A rigorous proof that finite and infinite sets in the sense of Cantor and Dedekind are the same requires (a version of) the Axiom of Choice. Again, while a formal treatment of the topic is beyond the scope of the projecct, students' understanding of finite vs. infinite is greatly enriched by this discussion.

The project ends with the discussion of countable and uncountable sets in general, and with Cantor's diagonalization method in particular, which provides students with an understanding of why the set of real numbers is uncountable.

Typically, students have little to no difficulty understanding the material presented in the first part of the project. However, the material in the second part of the project requires instructor guidance. It is advisable to have a detailed class discussion on some of the excerpts from Cantor and Dedekind, as well as on the cardinality of a set and on countable and uncountable sets. In particular, the open-ended exercises about the Cantor-Bernstein theorem, the Cantor and Dedekind definitions of finite and infinite sets, and the Axiom of Choice provide the instructor with an opportunity to have a more detailed class discussion on these topics.

2.4 *Pascal's Treatise on the Arithmetical Triangle*: Mathematical Induction, Combinations, the Binomial Theorem and Fermat's Theorem



An introductory course on discrete mathematics or proofs typically includes mathematical induction, index notation, binomial coefficients, combination numbers, factorials, and some elementary number theory. All these are naturally melded in a multi-week project [2, 4] replacing textbook material, based on Blaise Pascal's *Treatise on the Arithmetical*

Triangle from the 1650s, which expounds the principle of mathematical induction and leads into combinatorics. Pascal begins by defining the numbers in "Pascal's triangle" in the now standard iterative manner, and then observes numerous patterns that he would like to claim continue indefinitely. To do so he offers a condition for their persistence, stated explicitly in the proof of his "Twelfth Consequence." This is the very first place in the literature where the principle of mathematical induction was enunciated so completely and generally. His initial Consequences (theorems) build up simple patterns, easing into mathematical induction via iteration and generalizable example. The Twelfth Consequence, in which induction as a general technique is explicitly verbalized, results in the modern factorial formula for combination numbers and binomial coefficients, allowing Pascal to proceed to probability and the binomial theorem.

After historical background and context, students begin by reading Pascal's highly verbal defining description of his triangle, entirely labeled using letters. Pascal had no index notation; it emerges via the project as a useful modern tool. Such a verbal approach, to a triangle many students think they are already familiar with, but lacking modern indexing notation, and geometrically tilted from the modern view, challenges students' skill at translating to modern mathematical descriptions, and places all students on the same unfamiliar footing. The idea of having students learn by being placed on unfamiliar footing is encompassed in the French *dépaysement*, a condition in which one must approach things from unaccustomed points of view, and pay great attention to subtleties. One might translate it as 'being thrown off guard,' 'being bewildered,' 'being taken out of one's element.' It is one of the great strengths of learning from primary historical sources, since they were often written in dramatically different context and time, thus providing a very distinct, and often extremely valuable, point of view. By viewing a topic from a point of view different from the standard modern one, a broader, but also deeper and more flexible, understanding is gained when combined with the modern. On the other hand, this approach does not mean choosing sources that are inherently difficult to read; Pascal's treatise is eminently readable by students today, rich in detail, written for the uninitiated.

The project has exercises throughout, relating to modern indexing notation, summation, and the adequacy of Pascal's proofs by iteration or generalizable example. His sequence of claims and proofs actually morphs naturally into the concept of proof by induction, allowing this proof concept to evolve in students' minds, rather than being thrust abstractly on them from nowhere, as in many modern textbooks. Because he has no indexing notation, Pascal's proofs must rely on generalizable example. Having students use modern index notation to make this precise in full generality allows students to develop powerful appreciation for the efficacy of index notation.

The crowning technical property of the triangle, which Pascal proves as his Twelfth Consequence, provides a formula for the ratio of consecutive numbers in any base (row) of the triangle. From this he can deduce an elegant closed formula for all the numbers, the basis for future applications. We ask students to translate Pascal's proof by generalizable example into a completely modern proof by mathematical induction. This is nontrivial, and even involves understanding aspects of proportionality that are largely unknown today. The following excerpt, from the middle of the project, exhibits selections from the primary source, and the nature of exercises for students, to illustrate the challenge as well as the pedagogical richness of source content for teaching core material.

Twelfth Consequence

In every arithmetical triangle, of two contiguous cells in the same base the upper is to the lower as the number of cells from the upper to the top of the base is to the number of cells from the lower to the bottom of the base, inclusive.

Let any two contiguous cells of the same base, E, C, be taken. I say that

E	:	C	::	2	:	3
the lower		the upper		because there are two cells from E to the bottom, namely E, H,		because there are three cells from C to the top, namely C, R, μ.

Although this proposition has an infinity of cases, I shall demonstrate it very briefly by supposing two lemmas:

The first, which is self-evident, that this proportion is found in the second base, for it is perfectly obvious that $\varphi : \sigma :: 1 : 1$;

The second, that if this proportion is found in any base, it will necessarily be found in the following base.

Whence it is apparent that it is necessarily in all the bases. For it is in the second base by the first lemma; therefore by the second lemma it is in the third base, therefore in the fourth, and to infinity.

It is only necessary therefore to demonstrate the second lemma as follows: If this proportion is found in any base, as, for example, in the fourth, $D\lambda$, that is, if $D : B :: 1 : 3$, and $B : \theta :: 2 : 2$, and $\theta : \lambda :: 3 : 1$, etc., I say the same proportion will be found in the following base, $H\mu$, and that, for example, $E : C :: 2 : 3$.

For $D : B :: 1 : 3$, by hypothesis.

Therefore $\underbrace{D + B}_{E} : B :: \underbrace{1 + 3}_{4} : 3$

$$E : B :: 4 : 3$$

Similarly $B : \theta :: 2 : 2$, by hypothesis

Therefore $\underbrace{B + \theta}_{C} : B :: \underbrace{2 + 2}_{4} : 2$

$$C : B :: 4 : 2$$

But $\quad B : E :: 3 : 4$

Therefore, by compounding the ratios, $C : E :: 3 : 2$. Q.E.D.

The proof is the same for all other bases, since it requires only that the proportion be found in the preceding base, and that each cell be equal to the cell before it together with the cell above it, which is everywhere the case.

Exercise 1. Pascal's Twelfth Consequence: the key to our modern factorial formula

1. Rewrite Pascal's Twelfth Consequence as a generalized modern formula, entirely in our $T_{i,j}$ terminology. Also verify its correctness in a couple of examples taken from his table in the initial definitions section.
2. Adapt Pascal's proof by example of his Twelfth Consequence into modern generalized form to prove the formula you obtained above. Use the principle of mathematical induction to create your proof.

From his Twelfth Consequence Pascal can develop a "formula" (essentially the modern factorial formula) for the numbers in the triangle, which can then be used in future work on combinatorics, probability, and algebra, and we guide students to read Pascal to carry this out.

The project continues on optionally (depending on time, desire, and course syllabus) to study counting combinations from Pascal's treatise, then to have students conjecture and prove Fermat's Little Theorem via induction using Pascal's formula above, and ends with an optional excursion into the RSA cryptosystem. This goes well beyond the historical source, showing how it can serve as a natural foundation for the development of important modern topics.

With Pascal's original text as a student's initial contact with mathematical induction, a textbook could become a supplement or grist for comparison. We find that students come to grips with induction better through Pascal easing it in through iteration and generalizable example for simple patterns in the triangle, then formalizing and applying it further. At that point leading a class discussion comparing it with a more modern formulation can be productive. We quite intentionally wait on asking students to read a modern approach until they are comfortable with Pascal's. Indeed, many students remain more comfortable with Pascal's more verbal approach than with that of a more modern textbook. We tested this preference by setting a final exam question in which each student chose either a proof by induction of a standard homework-like summation formula, or to digest, explain, and adapt into a modern induction proof a Consequence (theorem) in Pascal's treatise that they had never seen before. Half the students chose to do new interpretation and modern proof work from the Pascal treatise on the final exam, and did it well!

3 CONCLUSION

We have highlighted four projects encompassing an entire introductory discrete mathematics course. The projects all follow our guided reading paradigm of primary source excerpts surrounded by historical context, biography, and mathematically unifying modern text supplied by us, punctuated by interruptive commentary and exercises with intentional pedagogical goals based on the primary source material.

Nonetheless, there are large differences in the roles played by primary source material between the projects. In "Deduction Through the Ages" and "An Introduction to Elementary Set Theory" we use sources by different authors to have students learn the same concepts from different perspectives. In the first project, these sources are spread over an immense span of time, presenting a grand evolution of the mathematical conception of deduction, while in the second project the authors were contemporaries and colleagues, but presented quite different ideas about how to approach the concept of infinity. "An Introduction to Symbolic Logic" is different in that the primary source excerpt is used only to give a clear explanation of the main concepts and their connection to each other. Finally, "Pascal's Treatise on the Arithmetical Triangle: Mathematical Induction, Combinations, the Binomial Theorem, and Fermat's Theorem" is different yet again, in that students study substantial portions of a single key source throughout, punctuated by numerous sets of exercises, and a little commentary, together providing guided engaged learning of a number of different topics.

Thus there is clearly great freedom for innovative and flexible implementation of the pedagogy through both the choice and use of primary source excerpts and guiding text and exercises, constrained only by the historical development of mathematical ideas in the human mind, as reflected in the primary sources. The disparate roles for primary sources in different projects are guided by the pedagogical goals and proclivities of the instructor, along with the nature of the mathematical topic and its primary source material.

Our team has developed more primary source projects for various topics in discrete mathematics and beyond, at both lower and upper division levels. For instance, we have projects on algorithms, discrete summation, combinatorics, graph theory, Boolean algebra, mathematical logic, and computer science [1, 2, 3, 4, 5, 20]. An instructor's guide that accompanies each of our projects provides information on the content goals of that project and suggestions for its implementation in the classroom. Instructors of an introductory discrete mathematics course can thus build their course around an entirely different set of projects, tailored to address their personal content goals for that course. Those who wish to further adapt a particular project to their course goals are able to do so by revising, adding or deleting exercises to the LaTeX file (available via our web resources [2, 3, 5]).

Finally, materials and information for our teaching of group theory, number theory, combinatorics, and mathematical logic entirely from primary historical sources can be found in [1, 3, 5, 8, 25]. The dream that all mathematics might be taught and learned through direct student engagement with primary historical sources is discussed in [24].

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