Notation: Let \( (R, \mathfrak{m}, k) \) be a Noetherian local ring of dimension \( d \).

1. Let \( R = k[[x, y]]/(x^2 - y^3) \). Compute \( \lambda(R/\mathfrak{m}^n) \) for every \( n \in \mathbb{N} \) and use these values to compute \( e(R) \).

2. Show that if \( R \) is a regular local ring, and \( f \in \mathfrak{m} \), then \( e(R/(f)) = \text{ord}(f) \).

\[ \text{Definition: } \text{ord}(f) = \max\{n \mid f \in \mathfrak{m}^n\}. \]

\[ \text{Hint: } \text{Set } t = \text{ord}(f). \text{ Show that } (f) \cap \mathfrak{m}^n = f\mathfrak{m}^{n-t} \text{ for every } n; \]

\[ \lambda(\mathfrak{m}^n/(\mathfrak{m}^{n+1} + f\mathfrak{m}^{n-t})) = \lambda(\mathfrak{m}^n/\mathfrak{m}^{n+1}) - \lambda(\mathfrak{m}^{n-t}/\mathfrak{m}^{n-t+1}). \]

3. Let \( I \) and \( J \) be \( \mathfrak{m} \)-primary \( R \)-ideals. Show
   (a) If \( d = 0 \), then \( e(I) = \lambda(R) \).
   (b) For every integer \( r \geq 1 \), we have \( e(I^r) = r^d e(I) \).
   (c) If \( J \subseteq I \), then \( e(J) \geq e(I) \).
   (d) Assume \( R = k[[x_1, \ldots, x_d]] \) and \( I \) is generated by homogeneous forms, then \( e(I) \leq e(\text{in}(I)) \) for any term order \( < \).

\[ \text{Hint: } \text{We have } \lambda(R/I) = \lambda(R/\text{in}(I)). \]

4. Let \( M \) be a finitely generated \( R \)-module and \( I \) an \( \mathfrak{m} \)-primary ideal. We define \( e_d(I, M) = \lim_{n \to \infty} \frac{\lambda(M/\mathfrak{m}^nM)}{\mathfrak{m}^n} \), the Hilbert-Samuel multiplicity of \( I \) on \( M \). Show that \( e_d(I, -) \) is additive in short exact sequences, i.e., if \( 0 \to M' \to M \to M'' \to 0 \) is exact, then \( e_d(I, M) = e_d(I, M') + e_d(I, M'') \).

\[ \text{Hint: } \text{By the Artin-Rees Lemma there exists } r \in \mathbb{N} \text{ such that } I^nM \cap M' = I^{n-r}(I^r M \cap M') \text{ for every } n \geq r. \]

5. (Associativity formula for multiplicities) Let \( M \) be a finitely generated \( R \)-module and \( I \) an \( \mathfrak{m} \)-primary ideal. Let \( \text{Ass}(R) \) be the set of prime ideals \( \mathfrak{p} \) such that \( \dim(R/\mathfrak{p}) = d \). Show that \( e_d(I, M) = \sum_{\mathfrak{p} \in \text{Ass}(R)} e_d(I, R/\mathfrak{p}) \lambda(M_\mathfrak{p}) \).

\[ \text{Hint: } \text{There exists a filtration } 0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M \text{ such that for every } i \text{ we have } M_i/M_{i+1} \cong R/\mathfrak{p} \text{ for some } \mathfrak{p} \in \text{Spec}(R). \]

6. Let \( M \) be a finitely generated module having a rank, i.e., there exists \( r \in \mathbb{N} \) such that \( M_\mathfrak{p} \cong R^r_\mathfrak{p} \) for every \( \mathfrak{p} \in \text{Ass}(R) \) (for example, \( M \) is any module over a domain \( R \)) and \( I \) is an \( \mathfrak{m} \)-primary ideal. Show that \( e_d(I, M) = e(I) \text{rank}_R(M) \).
Definition: Let $Q$ be total ring of fractions of $R$, then $\text{rank}_R(M)$ is defined as $\text{rank}_Q(M \otimes_R Q)$.

(7) Let $I$ be a square-free monomial ideal. Show that $e(R/I)$ is equal to the number of prime ideals in $\text{Assh}(R/I)$.

(8) The goal of this problem is to prove the following theorem due to Serre.

Theorem. Let $x_1, \ldots, x_d$ be a system of parameters of $R$ and $I = (x_1, \ldots, x_d)$. Then, $e(I) = \lambda(R/I)$ if and only if $R$ is Cohen-Macaulay.

We divide the proof in the following steps.

(a) Assume $R$ is Cohen-Macaulay, then $\text{gr}(I) = \bigoplus_{n \in \mathbb{N}} I^n/I^{n+1}$, the associated graded ring of $I$, is a polynomial ring over $R/I$. Conclude that $\lambda(R/I^n) = \binom{n+d-1}{d} \lambda(R/I)$ and then $e(I) = \lambda(R/I)$.

(b) Assume now that $e(I) = \lambda(R/I)$. Let $S = R/I[X_1, \ldots, X_d]$ and $\varphi : S \to \text{gr}(I)$ be the epimorphism given by $\varphi(X_i) = x_i + I^2 \in [\text{gr}(I)]_1$. Set $K = \ker \varphi$ which is a homogeneous $S$-ideal. Prove that $\lambda([K]_n)$ coincides with a polynomial of degree $d-2$ for $n \gg 0$.

(c) Assume there exists $m$ such that $[K]_m \neq 0$. Let $F \neq 0 \in [K]_m$ and $\mathcal{M}_n$ the set of monomials of degree $n$ in the variables $X_1, \ldots, X_d$. Show that set $\{F\alpha \mid \alpha \in \mathcal{M}_n\}$ is linearly independent (with coefficients in $R/I$).

(d) Obtain a contradiction from (b) and (c). What can you conclude?

(e) Show that $x_1, \ldots, x_d$ is a regular sequence, and then $R$ is Cohen-Macaulay.

(9) Show that if $I$ is an $m$-primary ideal then $\ell(I) = d$.

(10) Assume $R$ is Cohen-Macaulay and $|k| = \infty$. Let $I$ be an $m$-primary ideal and $J$ a minimal reduction of $I$, then $e(I) = \lambda(R/J)$.

(11) (a) Let $R = k[[x, y, z]]$ and $I = (x^a, y^b, z^c, xyz)$ where $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1$. Compute $e(I)$.

(b) More generally, assume $R = k[[x_1, \ldots, x_d]]$ and $I = (x_1^{a_1}, \ldots, x_d^{a_d}, x_1 \cdots x_d)$, where $\frac{1}{a_1} + \cdots + \frac{1}{a_d} \leq 1$. Compute $e(I)$.

(12) Let $I$ be an $m$-primary ideal that contains a regular element. The Ratliff-Rush closure of $I$ is defined as $I^* = \bigcup_{n \geq 1} (I^{n+1} : I^n)$. Show that $I^*$ is the largest ideal containing $I$ such that $e_j(I^*) = e_j(I)$ for every $j = 0, \ldots, d$, i.e., $I^*$ is the $d$th-coefficient ideal of $I$.

Hint: There exists a regular element $a \in I$ such that $I^{n+1} : a = I^n$ for every $n \gg 0$. Show that $(I^*)^n = I^n$ for $n \gg 0$. Also, show that if $J$ is an $m$-primary ideal containing $I$ and such that $e_j(J) = e_j(I)$ for every $j = 0, \ldots, d$, then $I^n = J^n$ for every $n \gg 0$.

(13) Let $I$ and $J$ be $m$-primary ideals. Show that $e_0(I|J) = e(J)$ and $e_d(I|J) = e(I)$.
(14) Let $I$ and $J$ be $m$-primary ideals. Show that for positive integers $r, s$ one has

$$e(I^r J^s) = \sum_{k=0}^{d} \binom{d}{k} r^k s^{d-k} e_k(I \mid J).$$

(15) Let $I$ and $J$ be $m$-primary ideal and assume $d = 2$. Show $e_1(I|J)^2 = e(I)e(J)$ if and only if $I$ and $J$ are projectively equivalent, i.e., $\overline{I^m} = \overline{J^n}$ for some positive integers $m, n$.

**Hint:** $\frac{e_1(I|J)}{e(I)} = \frac{e(J)}{e_1(I|J)} = \frac{m}{n}$.

(16) Let $\overline{R} = R/(0 :_R I^\infty)$. Prove $j(I) = j(I \overline{R})$.

**Hint:** Show $I^n \cap (0 :_R I^\infty) = 0$ for $n \gg 0$.

**Theorem:** Assume $d \geq 2$ and $|k| = \infty$. Let $x$ be a general element in $I$ and $\overline{R} = R/(x)$. Then $j(I) = j(I \overline{R})$.

(17) Assume $|k| = \infty$. Show that for general elements $x_1, \ldots, x_d$ of $I$, one has

$$j(I) = \lambda \left( \frac{R}{(x_1, \ldots, x_{d-1}) : I^\infty + (x_d)} \right).$$

(18) Assume $R$ is a regular local ring with $d = 2$ and $\ell(I) = 2$.

(a) Show that there exists a nonzero $f \in R$ and an $m$-primary ideal $Q$ such that $I = fQ$.

(b) Show that $j(I) = e(Q) + e(Q(R/(f)))$.

(c) Verify this identity with $I = x(x,y) \subset k[[x,y]]$.

(19) (a) Let $R = k[[x,y]]$ and $I = (x^2, xy)$. Use the length formula for $j$-multiplicity (problem (17)) to compute $j(I)$.

(b) Let $R = k[[x,y,z]]$ and $I = (xy, xz, yz)$. Use the volume formula for $j$-multiplicity of monomial ideals to compute $j(I)$.

(20) (a) Given a short exact sequence $0 \to M' \to M \to M'' \to 0$, show

$$\lambda(H^0_m(M)) \leq \lambda(H^0_m(M')) + \lambda(H^0_m(M'')).$$

(b) Use part (a) to show that if $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n$ is a filtration, then

$$\lambda(H^0_m(M_n/M_0)) \leq \sum_{i=0}^{n-1} \lambda(H^0_m(M_{i+1}/M_i)).$$

(c) Use part (b) to conclude $\limsup_{n \to \infty} \frac{\lambda(H^0_m(R/I^n))}{n^d} < \infty$. 

3
(21) Let \( S = R[S_1] \) be a standard graded algebra over \( R \) and \( N \) a finitely generated \( S \)-module. Prove \( \dim H^0_m(N) \leq \dim N \) and that equality occurs if and only if \( \dim N/mN = \dim N \). Use this to conclude \( j(I) \neq 0 \) if and only if \( \ell(I) = d \).

(22) Find two ideals \( J \subset I \) in \( k[[x, y]] \) such that \( j(I) = j(J) \neq 0 \), but \( \bar{I} \neq \bar{J} \).

(23) Let \( Q \) be an \( m \)-primary ideal and \( f \) a non zero-divisor. Assume depth \( R \geq 2 \), show that \( \varepsilon(fQ) = e(Q) \).

(24) Prove the volume formula for \( j \)-multiplicity and \( \varepsilon \)-multiplicity of monomial ideals in \( k[[x, y]] \).

**Hint:** Express \( I \) as \( fJ \) where \( J \) is \( m \)-primary. Then use problems (18.b) and (23).