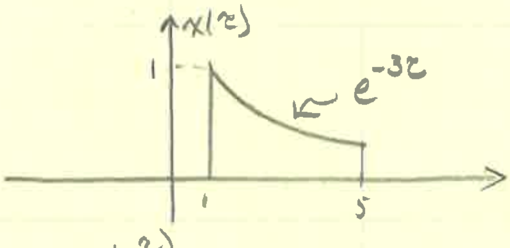
Continuous-Time

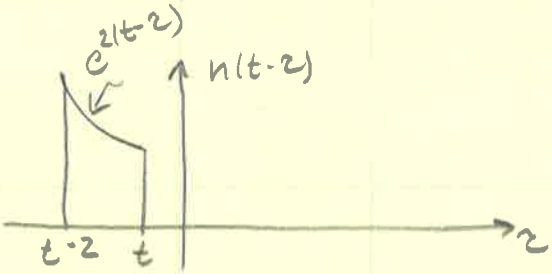
First, let's look at setting up a convolution problem involving two finite-length signals

Example 1c

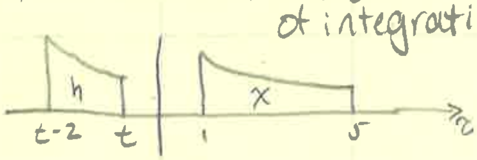
Let $x(t) = e^{-3t}[u(t-1) - u(t-5)]$, $h(t) = e^{2t}[u(t) - u(t-2)]$

Plot $x(\tau)$ and $h(t-\tau)$



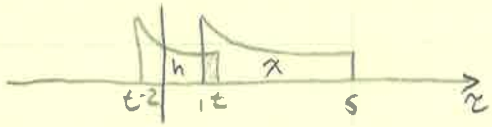


Now look at the different cases in terms of overlap and set limits of integration to reflect the region of overlap.



① $t < 1$ $\quad y(t) = 0$

② $1 < t < 3$ — @ $t=1$, overlap starts; @ $t-2=1$ overlap changes $\Rightarrow t=3$

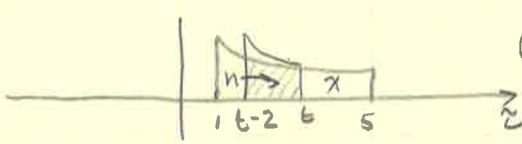


$$y(t) = \int_1^t e^{2(t-\tau)} e^{-3\tau}\,d\tau$$

$$= e^{2t}\int_1^t e^{(-5\tau)}\,d\tau = e^{2t}\cdot -\frac{1}{5}e^{-5\tau}\Big|_1^t$$

$$= e^{2t}\cdot -\frac{1}{5}\left(e^{-5t} - e^{-5}\right)$$

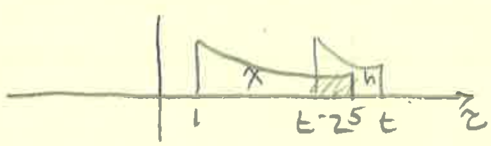
③ $3 < t < 5$ — @ $t=3$ from above; @ $t=5$ overlap changes (trailing edge of h in x)



$$y(t) = \int_{t-2}^t e^{2(t-\tau)} e^{-3\tau}\,d\tau$$

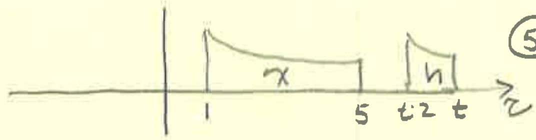
$$= -\frac{1}{5}e^{2t}e^{-5\tau}\Big|_{t-2}^t = -\frac{1}{5}e^{2t}\left(e^{-5t} - e^{-5(t-2)}\right)$$

④ $5 < t < 7$ — $t=5$ from above; @ $t-2=5$ overlap changes $\Rightarrow t=7$ (leading edge of h out of x)



$$y(t) = \int_{t-2}^5 e^{2(t-\tau)} e^{-3\tau}\,d\tau$$

$$= -\frac{1}{5}e^{2t}e^{-5\tau}\Big|_{t-2}^5 = -\frac{1}{5}e^{2t}\left(e^{-25} - e^{-5(t-2)}\right)$$



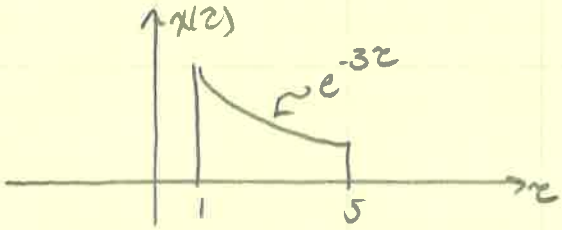
⑤ $t > 7 \quad y(t) = 0$

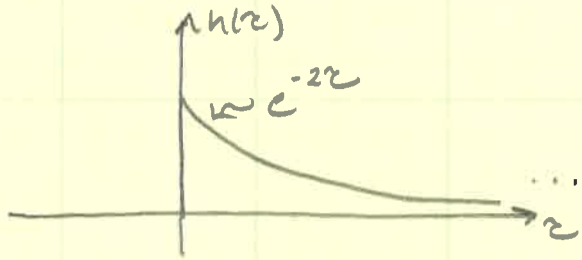
Now lets look at a similar case where one of the signals is infinite in length.

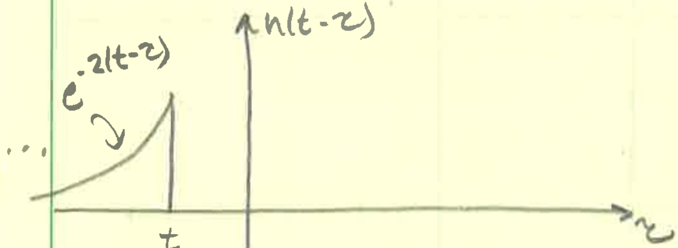
Example 2C

Let $x(t) = e^{-3t}[u(t-1) - u(t-5)]$, $h(t) = e^{-2t}u(t)$

Plot $x(\tau)$ and $h(t-\tau)$

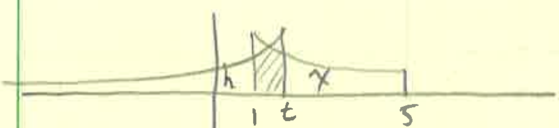






Look at different cases of overlap & set limits of integration to reflect region of overlap

① $t < 1 \quad y(t) = 0$

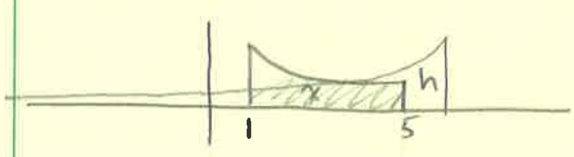


② $1 < t < 5$ — @ $t=1$ overlap begins
@ $t=5$ overlap changes (x entirely contained in h)

$$y(t) = \int_1^t e^{-2(t-\tau)} e^{-3\tau} \, d\tau$$

$$= e^{-2t}\int_1^t e^{-\tau} \, d\tau = -e^{-2t}e^{-\tau}\Big|_1^t$$

$$= -e^{-2t}\left(e^{-t} - e^{-1}\right)$$



③ $t > 5$

$$y(t) = \int_1^5 e^{-2(t-\tau)} e^{-3\tau} \, d\tau$$

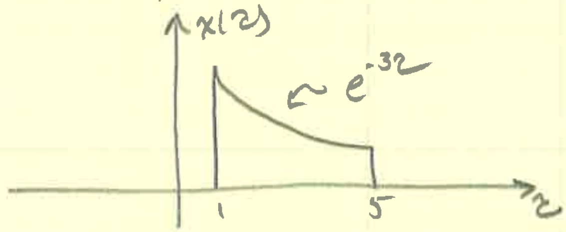
$$= -e^{-2t}e^{-\tau}\Big|_1^5 = -e^{-2t}\left(e^{-5} - e^{-1}\right)$$

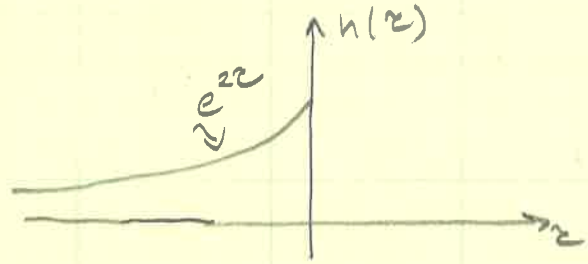
What if $h(t) = e^{2t}u(-t)$ in the previous example?

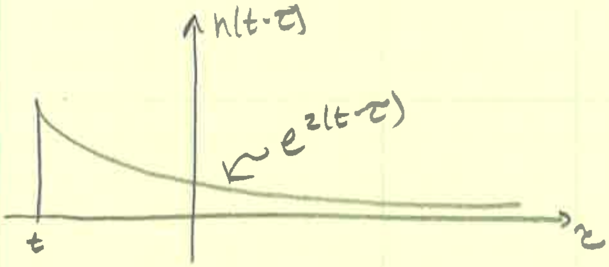
Example 3C

Let $x(t) = e^{-3t}[u(t-1) - u(t-5)]$, $h(t) = e^{2t}u(-t)$

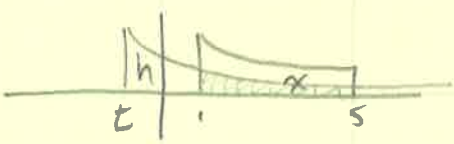
Plot $x(\tau)$ and $h(t-\tau)$







Look at different cases of overlap and set limits of integration accordingly



① $t < 1$

$$y(t) = \int_1^5 e^{2(t-\tau)} e^{-3\tau} d\tau$$

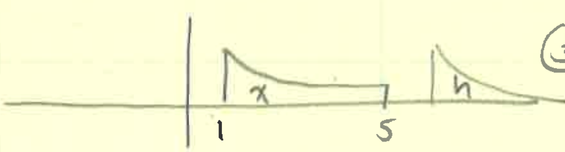
$$= e^{2t}\int_1^5 e^{-5\tau} d\tau$$

$$= -\frac{1}{5}e^{2t}e^{-5\tau}\Big|_1^5 = -\frac{1}{5}e^{2t}\left(e^{-25} - e^{-5}\right)$$

② $1 < t < 5$ @ $t=1$ overlap changes (trailing edge of h reaches x)
@ $t=5$ overlap ends

$$y(t) = \int_t^5 e^{2(t-\tau)} e^{-3\tau} d\tau$$

$$= -\frac{1}{5}e^{2t}e^{-5\tau}\Big|_t^5 = -\frac{1}{5}e^{2t}\left(e^{-25} - e^{-5t}\right)$$



③ $t > 5$ $y(t) = 0$

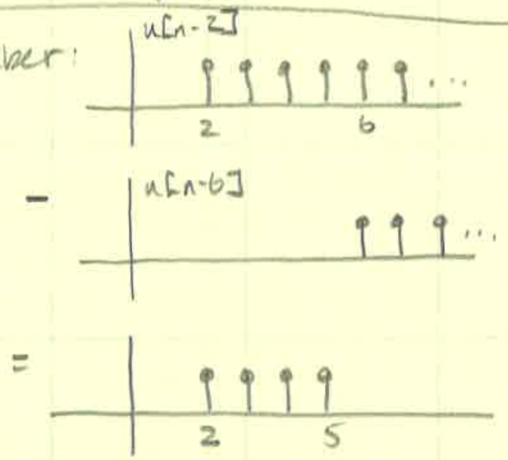
Discrete-Time

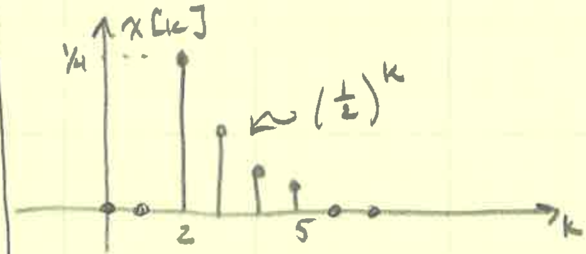
First, let's take the case of 2 finite-length signals

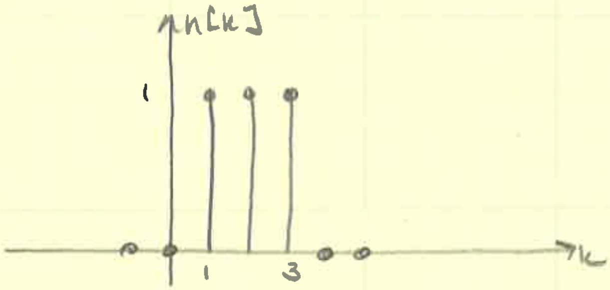
Example 1D

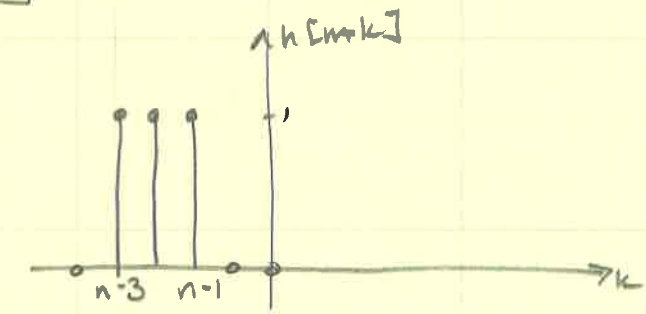
Let $x[n] = \left(\frac{1}{2}\right)^n \left(u[n-2] - u[n-6]\right)$, $h[n] = u[n-1] - u[n-4]$

remember:

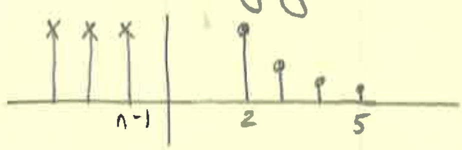








Look at different cases of overlap & set limits of summation accordingly (or use table method)



① $n-1 < 2 \Rightarrow n < 3 \quad y[n] = 0$

② $2 \le n \le 5$

n=2 from previous region @ n−3=2

overlap changes ⇒ n=5 (trailing edge of h reaches x)

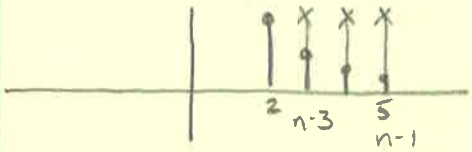
$$y[n] = \sum_{k=2}^{n-1} \left(\frac{1}{2}\right)^k \cdot 1 \qquad \text{let } m = k-2, \quad k = m+2$$

$$= \sum_{m=0}^{n-3} \left(\frac{1}{2}\right)^{m+2}$$

$$= \left(\frac{1}{2}\right)^2 \sum_{m=0}^{n-3} \left(\frac{1}{2}\right)^m$$

$$= \left(\frac{1}{2}\right)^2 \frac{1-\left(\frac{1}{2}\right)^{n-2}}{1-\frac{1}{2}}$$

$$= \frac{1}{4} \cdot 2 \cdot \left(1 - \left(\frac{1}{2}\right)^{n-2}\right) = \frac{1}{2}\left[1 - \left(\frac{1}{2}\right)^{n-2}\right]$$



③ $5 \le n \le 6$ — @ $n=6$ ($n-1=5$) overlap changes (leading edge of h not overlapping)

$$y[n] = \sum_{k=n-3}^{n-1} \left(\tfrac{1}{2}\right)^k \qquad \text{let } m = k-n+3, \quad k = m+n-3$$

$$= \sum_{m=0}^{2} \left(\tfrac{1}{2}\right)^{m+n-3}$$

$$= \left(\tfrac{1}{2}\right)^{n-3} \sum_{m=0}^{2} \left(\tfrac{1}{2}\right)^m$$

$$= \left(\tfrac{1}{2}\right)^{n-3} \cdot \frac{1-\left(\frac{1}{2}\right)^3}{1-\frac{1}{2}}$$

$$= 2\cdot\left(\tfrac{1}{2}\right)^{n-3}\left(1-\left(\tfrac{1}{2}\right)^3\right)$$

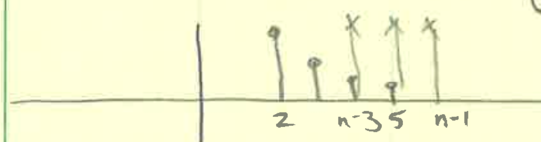
a quick check that summations were computed properly

sanity check @ $n=5$

$y[n]$ from ② $= \frac{1}{2}\left[1-\left(\frac{1}{2}\right)^3\right]$

$y[n]$ from ③ $= 2\cdot\left(\frac{1}{2}\right)^2\left[1-\left(\frac{1}{2}\right)^3\right]$

same ✓



④ $6 \le n \le 8$ — @ $n-3=5 \Rightarrow n=8$ overlap changes (no more overlap)

$$y[n] = \sum_{k=n-3}^{5} \left(\tfrac{1}{2}\right)^k \qquad \text{let } m = k-n+3, \quad k = m+n-3$$

$$= \sum_{m=0}^{8-n} \left(\tfrac{1}{2}\right)^{m+n-3}$$

$$= \left(\tfrac{1}{2}\right)^{n-3} \sum_{m=0}^{8-n} \left(\tfrac{1}{2}\right)^m$$

$$= \left(\tfrac{1}{2}\right)^{n-3} \cdot \frac{1-\left(\frac{1}{2}\right)^{9-n}}{1-\frac{1}{2}}$$

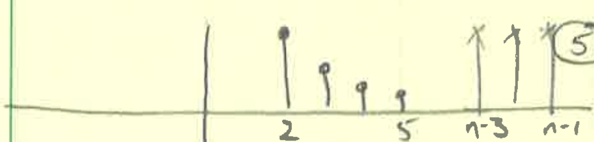
$$= 2\cdot\left(\tfrac{1}{2}\right)^{n-3}\left[1-\left(\tfrac{1}{2}\right)^{9-n}\right]$$

sanity check @ $n=6$

$y[n]$ from ③ $= 2\left(\frac{1}{2}\right)^3\left[1-\left(\frac{1}{2}\right)^3\right]$

④ $= 2\left(\frac{1}{2}\right)^3\left[1-\left(\frac{1}{2}\right)^3\right]$

same ✓



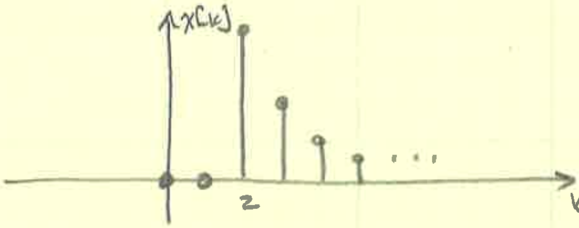
⑤ $n > 8 \quad y[n] = 0$

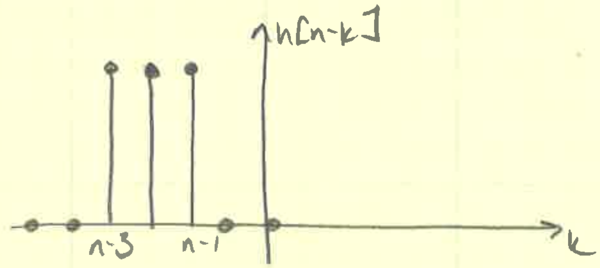
Similar case w/ one infinite length signal.

<u>Example 2D</u>

Let $x[n] = \left(\frac{1}{2}\right)^n u[n-2]$, $h[n] = u[n-1] - u[n-4]$

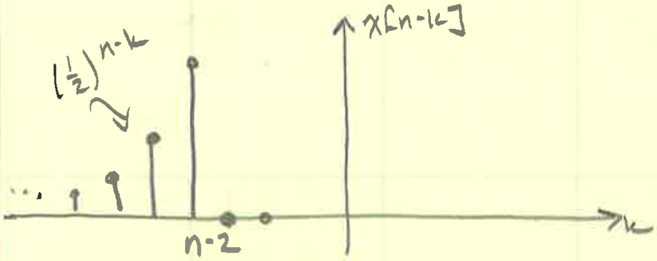
Plot $x[k]$ and $h[n-k]$



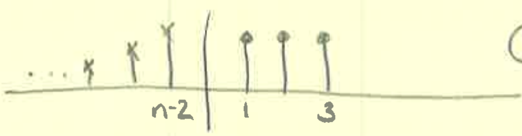


Actually, lets do it the other way

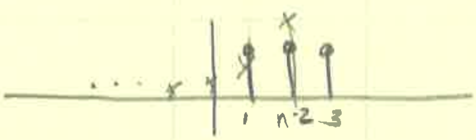
Plot $x[n-k]$ and $h[k]$



Look for different cases of overlap & set limits accordingly



① $n < 3$ $\quad y[n] = 0$



② $3 \le n \le 5$ – @ $n-2=3 \Rightarrow n=5$ overlap changes (h completely contained in x)

$$y[n] = \sum_{k=1}^{n-2} \left(\tfrac{1}{2}\right)^{n-k}$$

let $m = k-1$

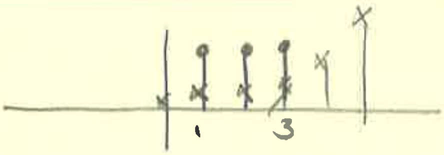
$k = m+1$

$$= \left(\tfrac{1}{2}\right)^n \sum_{m=0}^{n-3} \left(\tfrac{1}{2}\right)^{-(m+1)}$$

$$= \left(\tfrac{1}{2}\right)^n \left(\tfrac{1}{2}\right)^{-1} \sum_{m=0}^{n-3} \left(\tfrac{1}{2}\right)^{-m}$$

$$= 2 \cdot \left(\tfrac{1}{2}\right)^n \sum_{m=0}^{n-3} 2^m$$

$$= 2 \cdot \left(\tfrac{1}{2}\right)^n \frac{1-2^{n-2}}{1-2}$$



③ $n \geq 5$

$$y[n] = \sum_{k=1}^{3} \left(\tfrac{1}{2}\right)^{n-k} \qquad \text{let } m = k-1,\quad k = m+1$$

$$= \left(\tfrac{1}{2}\right)^{n} \sum_{m=0}^{2} \left(\tfrac{1}{2}\right)^{-(m+1)}$$

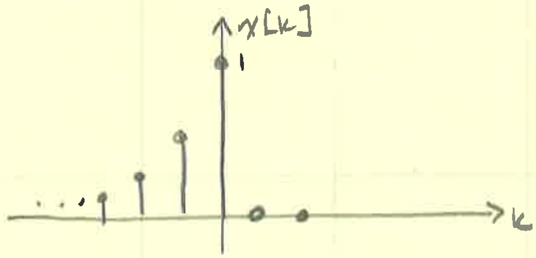
$$= \left(\tfrac{1}{2}\right)^{n} \left(\tfrac{1}{2}\right)^{-1} \sum_{m=0}^{2} \left(\tfrac{1}{2}\right)^{-m}$$

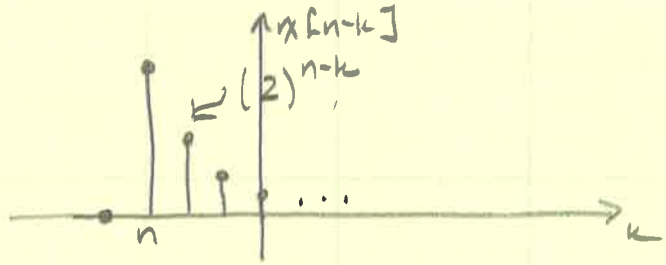
$$= \left(\tfrac{1}{2}\right)^{n} \cdot 2 \sum_{m=0}^{2} (2)^{m}$$

$$= \left(\tfrac{1}{2}\right)^{n} \cdot 2 \cdot \frac{1-2^3}{1-2}$$

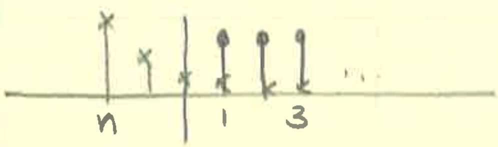
Example 3D

Let $x[n] = (2)^n u[-n]$, $h[n] = u[n-1] - u[n-4]$





$h[k]$ same plot as in 2D



① $n \leq 1$

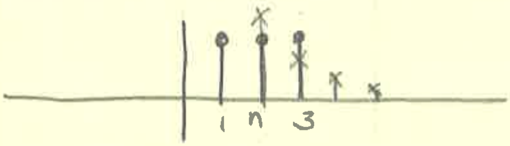
$$y[n] = \sum_{k=1}^{3} (2)^{n-k} \qquad \text{let } m = k-1,\quad k = m+1$$

$$= 2^n \sum_{m=0}^{2} 2^{-(m+1)}$$

$$= 2^n\, 2^{-1} \sum_{m=0}^{2} \left(\tfrac{1}{2}\right)^{m}$$

$$= 2^n \cdot \cancel{\tfrac{1}{2}} \cdot \frac{1-\left(\frac{1}{2}\right)^3}{\cancel{1-\frac{1}{2}}}$$

$$= 2^n \left(1 - \left(\tfrac{1}{2}\right)^3\right)$$



(2) $1 \leq n \leq 3$

$$y[n] = \sum_{k=n}^{3} (2)^{n-k} \qquad \text{let } m = k-n, \quad k = m+n$$

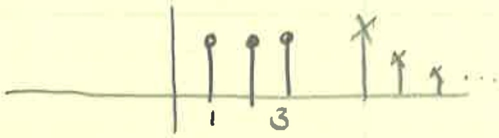
$$= 2^n \sum_{m=0}^{3-n} (2)^{-(m+n)}$$

$$= \cancel{2^n} \cdot \cancel{2^{-n}} \sum_{m=0}^{3-n} (2)^{-m}$$

$$= \sum_{m=0}^{3-n} \left(\tfrac{1}{2}\right)^m$$

$$= \frac{1 - \left(\frac{1}{2}\right)^{4-n}}{1 - \frac{1}{2}}$$

$$= 2 \cdot \left(1 - \left(\tfrac{1}{2}\right)^{4-n}\right)$$



(3) $n > 3 \quad y[n] = 0$