# Solving Linear Constant Coefficient Difference / Differential Equations

An LCCDE is an *implicit* description of a system; it describes a relationship between the input and output of a system. What we are generally interested in is an *explicit* description of a system - one that expresses the output as a function of the input. To find the *explicit* description, we have to solve the LCCDE.

## Continuous-Time (Differential)

We will only concern ourselves with first order LCCDEs for this class.

Example 2.14

Given a system described by

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

with $x(t) = Ke^{3t}u(t)$, what is $y(t)$?

We can solve this with 2 basic approaches

1) Solve the differential equation using the given $x(t)$ to find the particular solution
2) Find the impulse response of the system $h(t)$ and find $y(t)$ by convolution $y(t) = x(t) * h(t)$

1) Solve diffeq

Assume homogeneous solution of the form

$y_h(t) = Ae^{st}$

substitute into diffeq w/ $x(t) = 0$

$\cancel{A}se^{st} + 2\cancel{Ae^{st}} = 0$

$s + 2 = 0$

$s = -2 \quad \Rightarrow \quad y_h(t) = Ae^{-2t}$

We'll use initial conditions to solve for A later

Now assume a particular solution of the form

$y_p(t) = Bx(t) \quad , \quad t > 0$

$\quad = BKe^{3t} \quad , \quad t > 0$

substitute into diffeq w/ $x(t) = Ke^{3t}$

$3B\cancel{Ke^{3t}} + 2BKe^{3t} = \cancel{Ke^{3t}}$

$5B = 1$

$B = 1/5 \quad \Rightarrow \quad y_p(t) = \frac{K}{5}e^{3t} \quad t > 0$

Total solution is

$$y(t) = y_h(t) + y_p(t)$$
$$= \left(Ae^{-2t} + \tfrac{k}{5}e^{3t}\right) u(t)$$

← since $y_p(t)$ was valid only for $t>0$

assume causality (initial rest) $y(0) = 0$

$$y(0) = A + \tfrac{k}{5} = 0$$
$$A = \tfrac{-k}{5}$$

$$\boxed{y(t) = \left(-\tfrac{k}{5}e^{-2t} + \tfrac{k}{5}e^{3t}\right) u(t)}$$

2) Find impulse response & convolve

If we first found the step response $s(t)$, we know that the impulse response $h(t) = \frac{d}{dt}\, s(t)$

The step response of the system is the output for $x(t) = u(t)$
Think of this as $x(t) = e^{0t}u(t)$ if it makes it easier. Thus, we can solve for the step response in a similar manner as above

The homogeneous solution is the same

$$y_h(t) = Ae^{-2t}$$

Assume a particular solution of the form

$$y_p(t) = Bx(t), \quad t>0$$
$$= B, \quad t>0$$

Substitute into diffeq

$$0 + 2B = 1$$
$$B = 1/2$$

$$y(t) = Ae^{-2t} + \tfrac{1}{2} \qquad y(0) = A + \tfrac{1}{2} = 0$$
$$A = -1/2$$

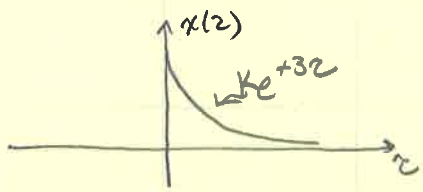
$$y(t) = \left(-\tfrac{1}{2}e^{-2t} + \tfrac{1}{2}\right) u(t)$$

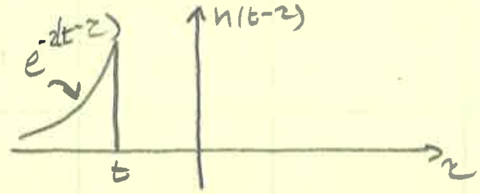
since $x(t) = u(t)$ this output is the step response $s(t)$

$$s(t) = \left(-\tfrac{1}{2}e^{-2t} + \tfrac{1}{2}\right) u(t)$$

$$\boxed{h(t) = e^{-2t}\, u(t)}$$

Now convolve $h(t)$ with $x(t) = Ke^{3t}u(t)$ to find output




① $t<0$, $y(t) = 0$

② $t>0$ $\quad y(t) = \int_0^t Ke^{+3\tau} e^{-2(t-\tau)}\,d\tau$

$$= Ke^{-2t}\int_0^t e^{+5\tau}\,d\tau = \frac{K}{5}e^{-2t}e^{5\tau}\Big|_0^t$$

$$= \frac{K}{5}e^{-2t}(e^{5t} - e^{0}) = \frac{K}{5}(e^{3t} - e^{-2t})$$

$\boxed{y(t) = \frac{K}{5}(e^{3t} - e^{-2t})\,u(t)}$ same as (1)

Discrete-Time (Difference)

Example 1.10

$$y[n] = 1.01\,y[n-1] + x[n] \quad , \quad x[n] = 100\,u[n]$$

Can solve this with the same basic 2 approaches as in CT
1) Solve for closed form solution using given $x[n]$
2) Find impulse response and convolve

1) You can solve the difference equation with a similar homogeneous / particular solution approach as in CT - refer to Problem 2.32 for more description of this

Or, you can look for a pattern in the output

* initial rest / causality condition

| n | x[n] | y[n-1] | y[n] |
|---|---|---|---|
| 0 | 100 | 0 * | 100 |
| 1 | 100 | 100 | 1.01(100) + 100 |
| 2 | 100 | 1.01(100) + 100 | 1.01(1.01(100) + 100) + 100 |
| 3 | 100 | | ⋮ |

$$y[n] = \sum_{k=0}^{n} (1.01)^k\,100 \qquad n > 0$$

$$= 100 \cdot \frac{1-(1.01)^{n+1}}{1-1.01}$$

It isn't always easy to see the pattern here.

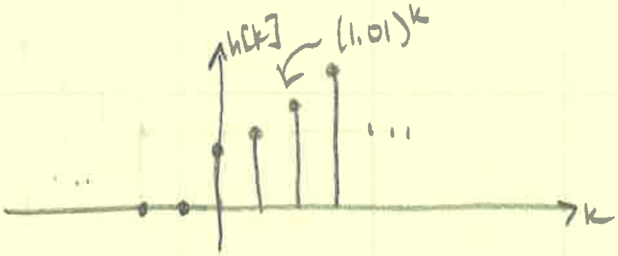
2) Find impulse response & convolve

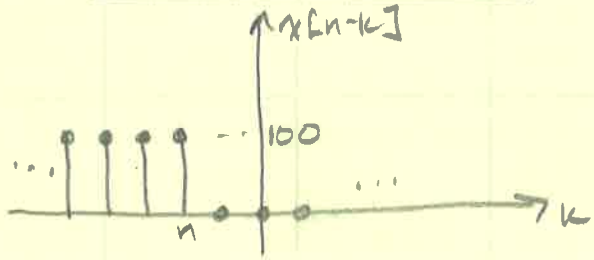
set $x[n] = \delta[n] \Rightarrow y[n] = h[n]$ and look for pattern

| $n$ | $x[n]=\delta[n]$ | $y[n-1]$ | $y[n]=h[n]$ |
|---|---|---|---|
| 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1.01 |
| 2 | 0 | 1.01 | $1.01^2$ |
| 3 | 0 | $1.01^2$ | $1.01^3$ |
| 4 | 0 | | |

often easier to see the pattern here

$$\boxed{h[n] = (1.01)^n u[n]}$$




① $n<0 \quad y[n]=0$

② $n>0 \quad y[n] = \sum_{k=0}^{n} (1.01)^k \cdot 100$

$$= 100 \cdot \frac{1-(1.01)^{n+1}}{1-1.01} \quad \text{same as (1)}$$